On classifying link maps in the 4-sphere

Ash Lightfoot, HSE

4th Russian-Chinese Conference on Knot Theory and Related Topics

July 3, 2017

Outline

1. Link Homotopy
2. Intersections of surfaces in a 4-manifold
3. Kirk's σ invariant of link homotopy
4. Techniques to address the open problem:

   does $\sigma = 0 \Rightarrow$ nullhomotopic?

The Classification Problem

Link ***map***:

$$f : S^{p_1} \cup S^{p_2} \cup \ldots \cup S^{p_n} \to S^m, \quad f(S^{p_i}) \cap f(S^{p_j}) = \varnothing$$

for $i \neq j$

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Problem: (For fixed p_i, n, m) Classify the set

$$\frac{\{\text{link maps } f : S^{p_1} \cup S^{p_2} \cup \ldots \cup S^{p_n} \to S^m\}}{\text{link homotopy}}$$

What do we know?

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classification $\longleftrightarrow$ homotopy theory questions

in certain dimension ranges

Hard: links maps in S^4

$$f : S^2_+ \cup S^2_- \to S^4, \qquad f(S^2_+) \cap f(S^2_-) = \varnothing$$

Write $f_+ = f|_{S^2_+}$, $f_- = f|_{S^2_-}$

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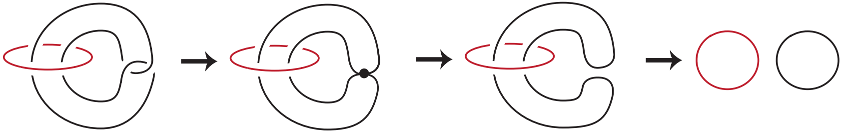

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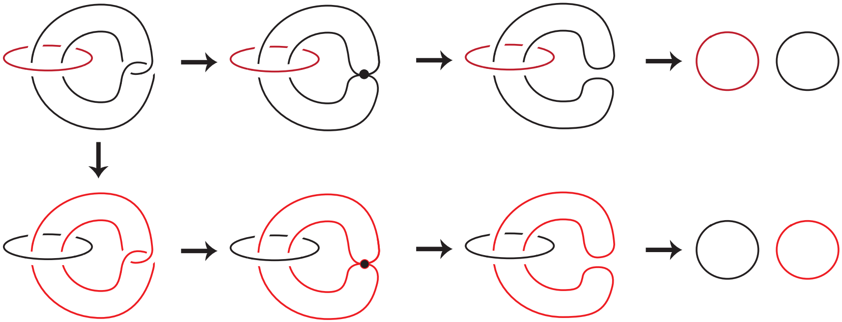

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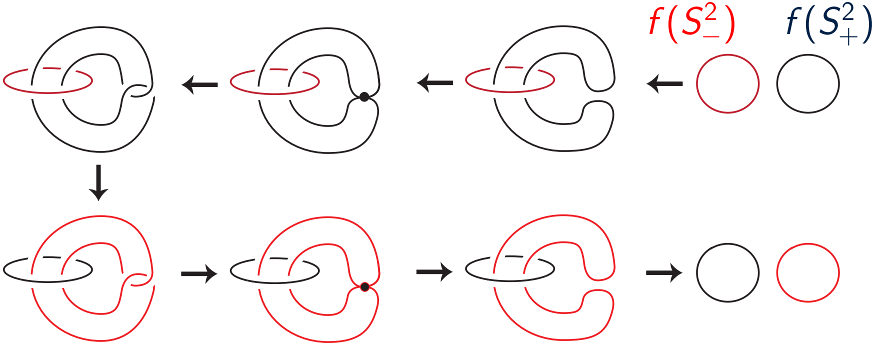

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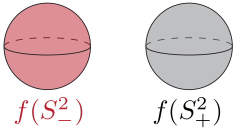
(Trivial link map: two embedded 2-spheres bounding disjoint 3-balls)



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Q: When is a link map link homotopic to ~~the trivial link?~~ an embedding? (Bartels-Teichner '99)

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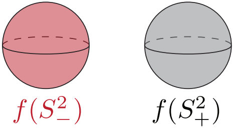
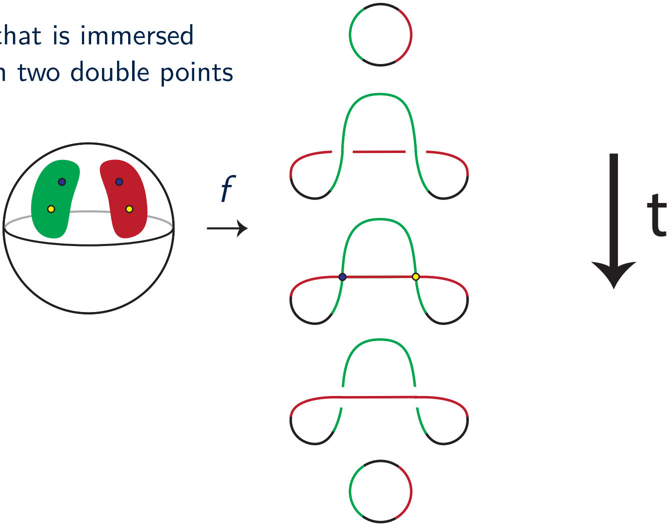


Self-intersections of a 2-sphere

Consider a simple map $f : S^2 \to \mathbb{R}^4$

- ... that is immersed with two double points



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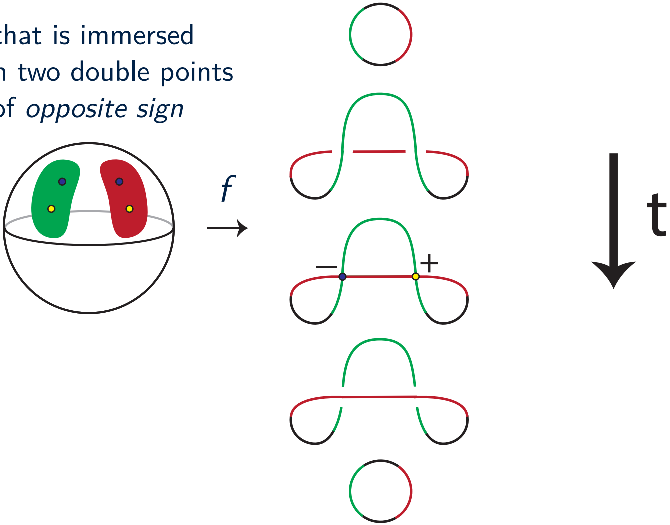

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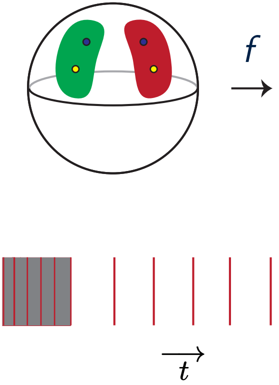

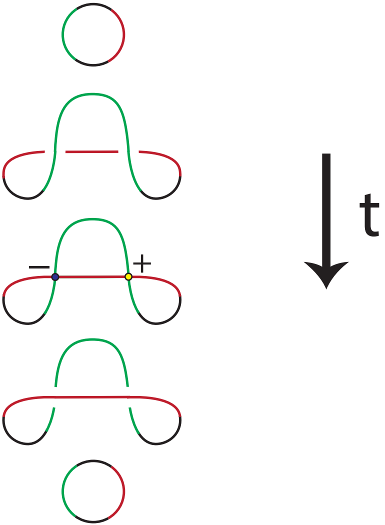

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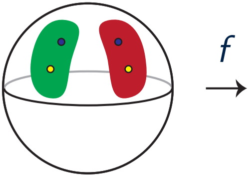

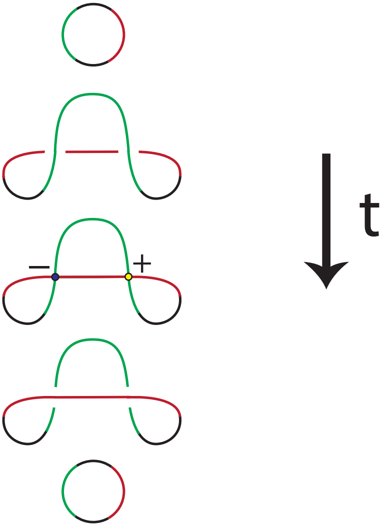

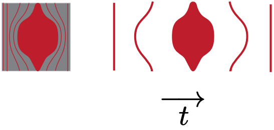

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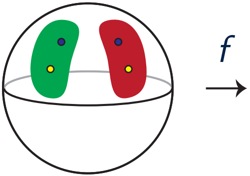

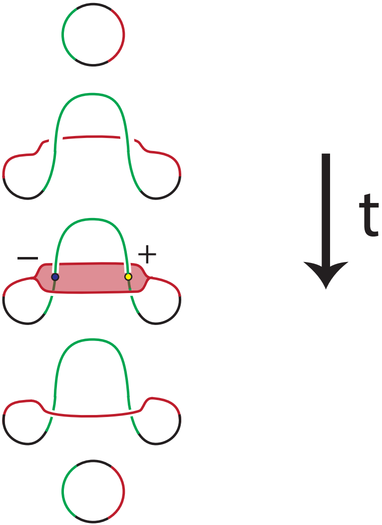

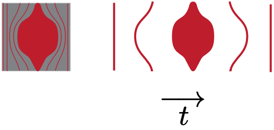

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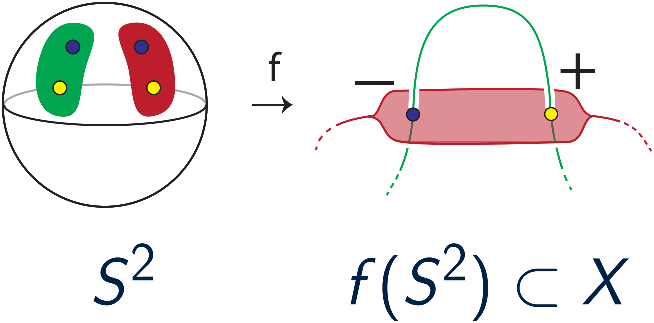

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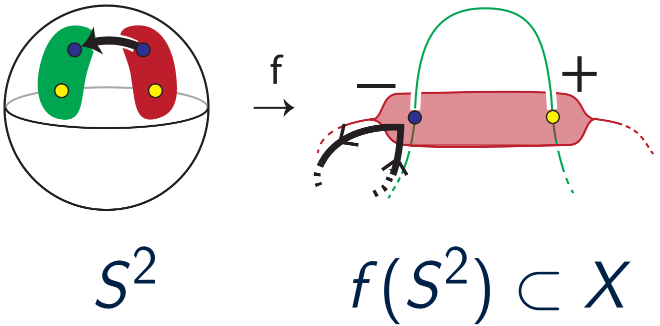

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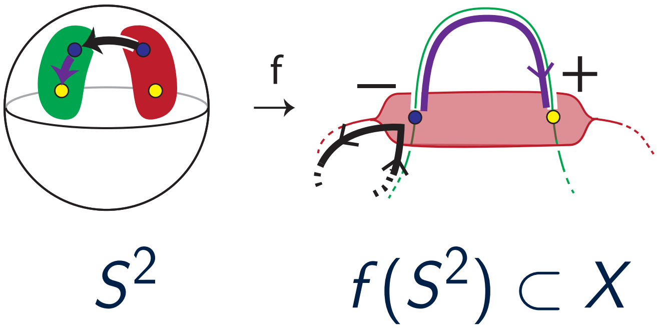

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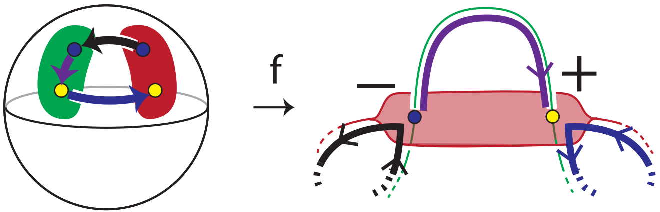

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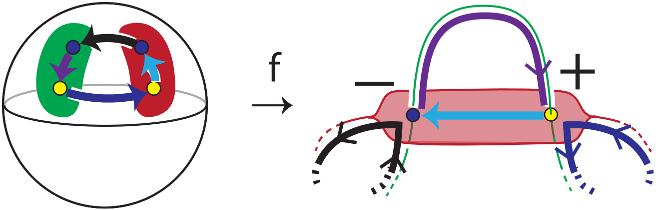

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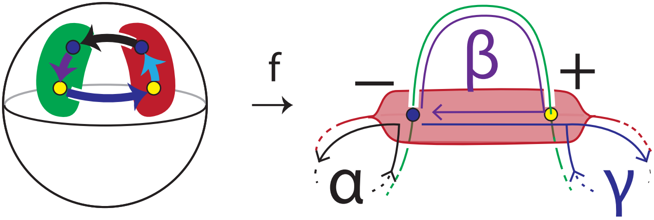

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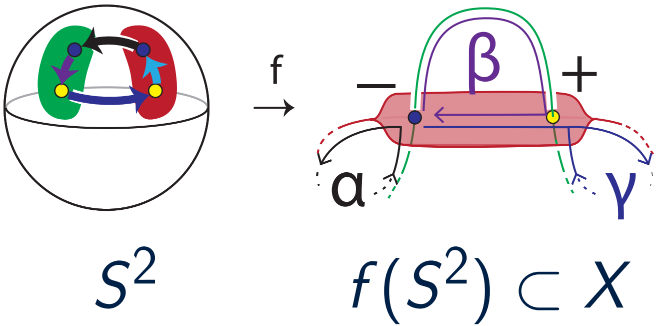

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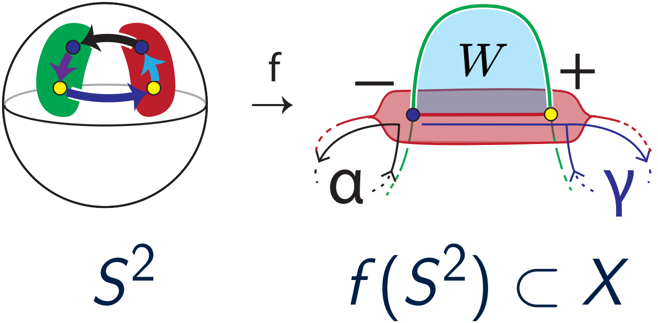

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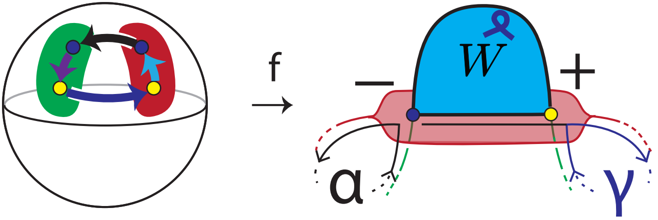

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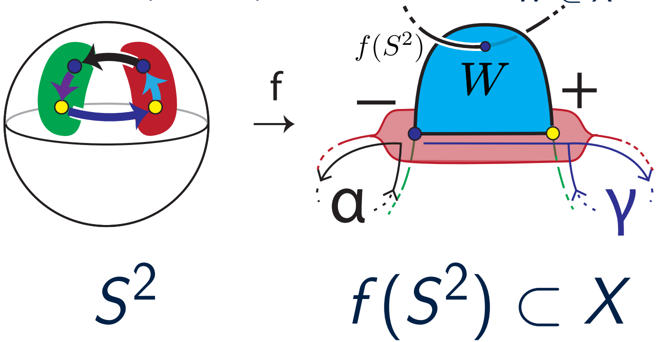

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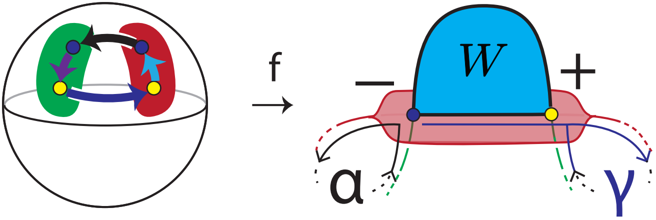

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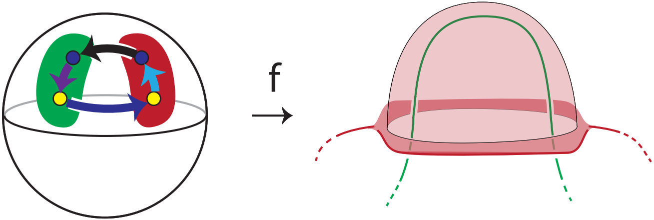

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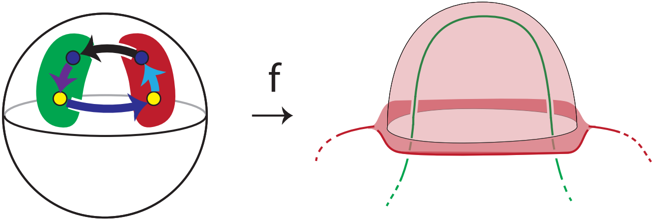

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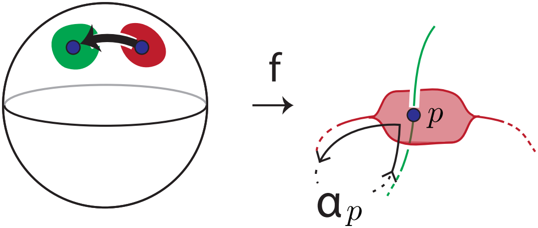

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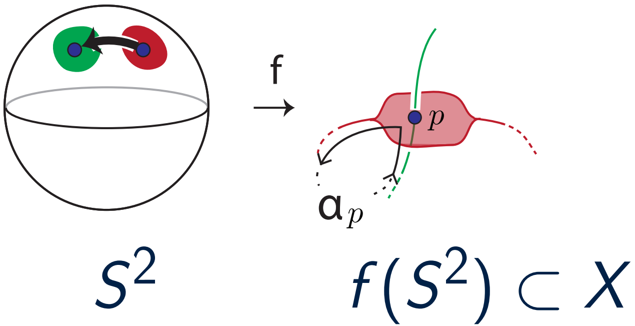

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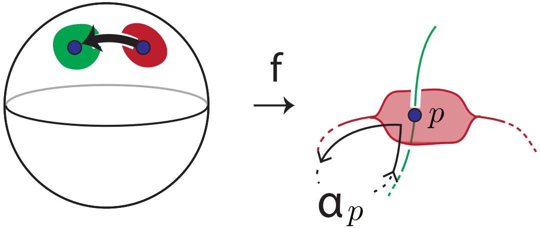

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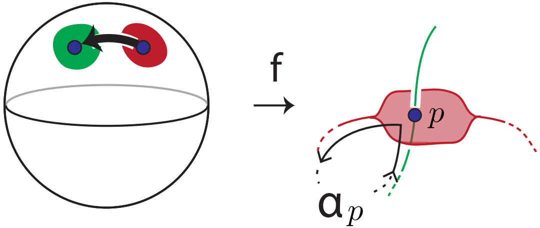

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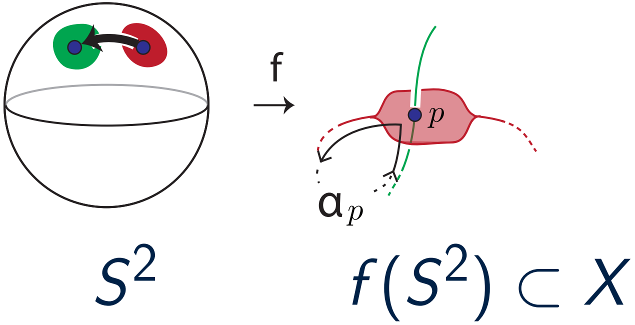

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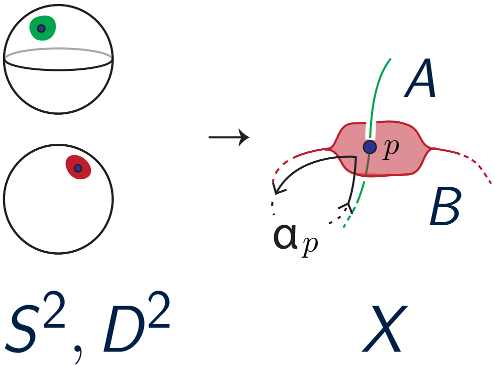

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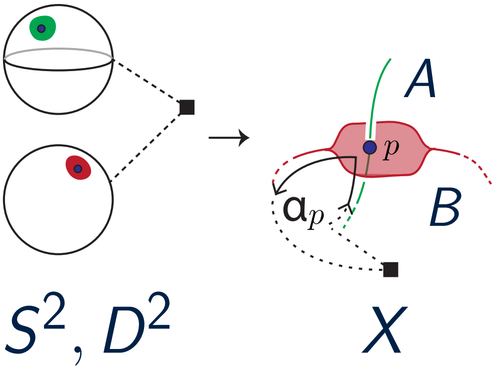

Kirk's link homotopy invariant σ

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$$\sigma_\pm(f) = \mu(f_\pm) = \sum_{p \in \mathrm{self}(f_\pm)} \mathrm{sign}_p(t^{n_p} - 1) \in \mathbb{Z}[t]$$

Kirk's link homotopy invariant $\sigma = (\sigma_+, \sigma_-)$

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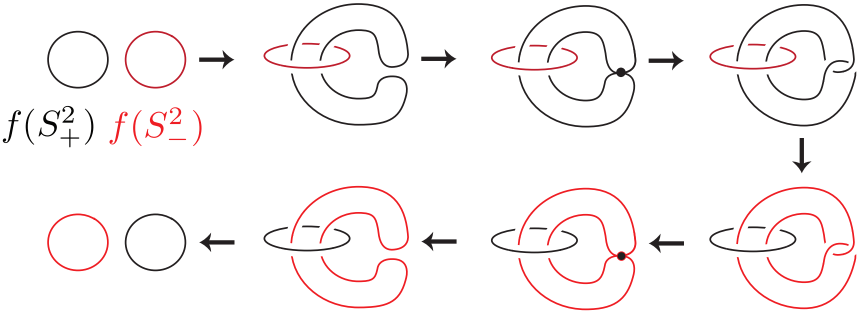

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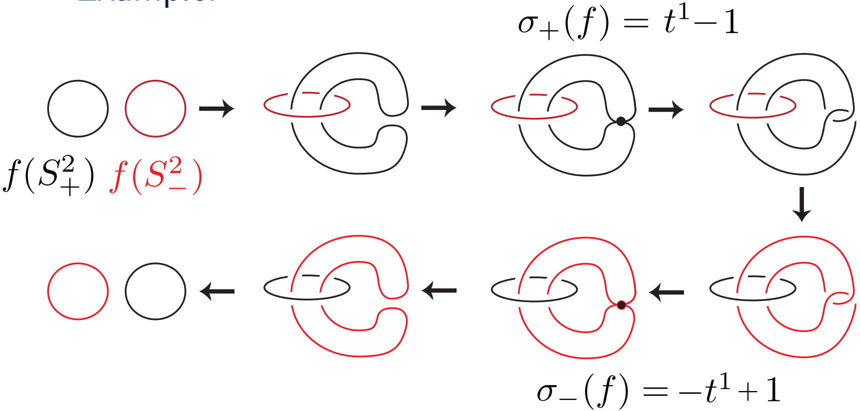


Properties of σ:

- Link homotopy invariant
- f link homotopic to embedding

  $\Rightarrow \sigma_+(f) = 0 = \sigma_-(f)$
- $\sigma_\pm(f) = 0$

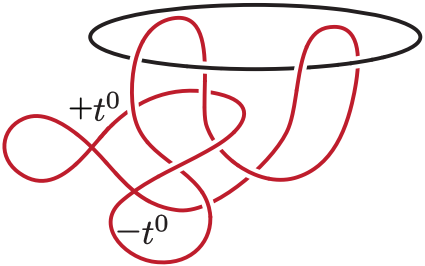
  $\Rightarrow$ can equip $f_\pm$ with Whitney disks in $S^4 \setminus f(S^2_\mp)$

Is σ the complete obstruction to embedding?

That is, is the existence of Whitney disks alone enough to embed?

$f(S^2_+)$

$f(S^2_-)$



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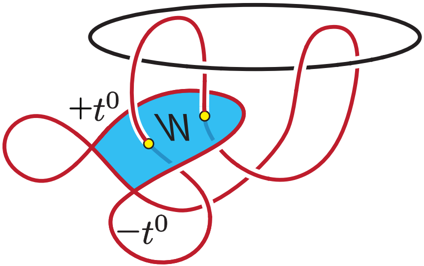

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The Whitney disk intersects $f(S^2_-)$...

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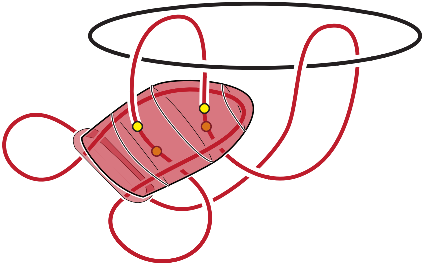

$\sigma_+(f) = -t^2 + 4t - 3$

$\sigma_-(f) = t^0 - t^0 = 0$

The Whitney disk intersects $f(S^2_-)$... so can't use to homotope f_- to an embedding

Is σ the complete obstruction to embedding?

That is, is the existence of Whitney disks alone enough to embed?

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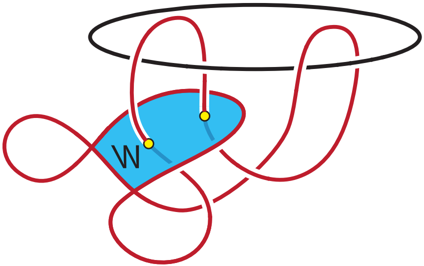

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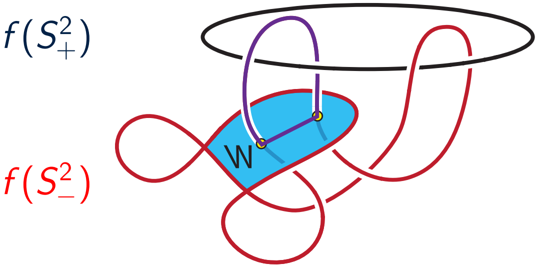

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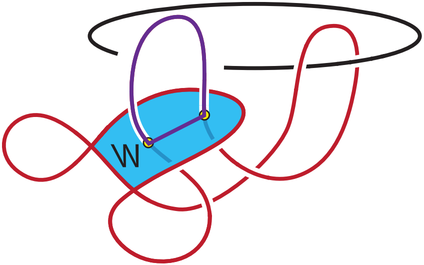

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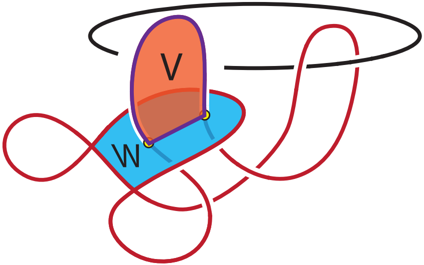

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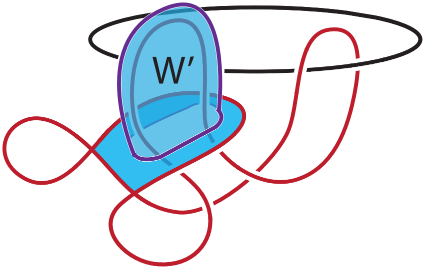

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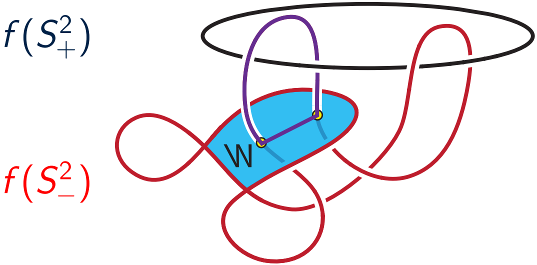

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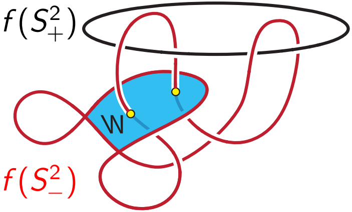
$\rightsquigarrow$ define a “secondary” invariant that obstructs this

Is σ the complete obstruction to embedding?

Some history:

- 1997: Li defined a secondary link htpy invariant $\omega = (\omega_+, \omega_-)$



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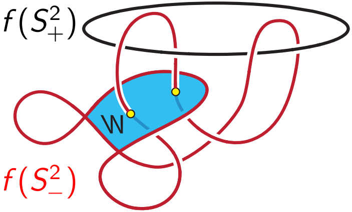

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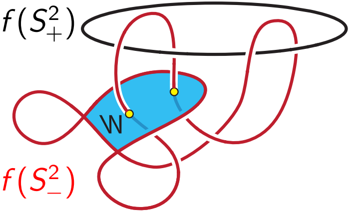

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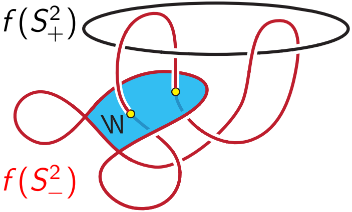

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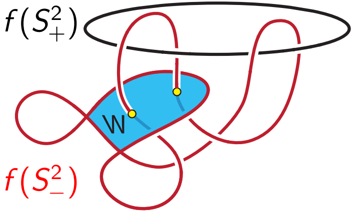
- 1997: Pilz found *mistake* in Li's example (actually had $\omega = (0,0)$)

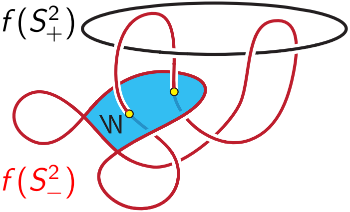


Nothing new: $\sigma(f) = (0,0) \Rightarrow \omega(f) = (0,0)$

Theorem (L.)

*If $f : S^2_+ \cup S^2_- \to S^4$ is a link map with **both** $\sigma_+(f) = 0$ and $\sigma_-(f) = 0$, then:*

(after a link homotopy) each component $f_\pm$ can be equipped with framed, immersed Whitney disks whose interiors are disjoint from both $f(S^2_+)$ and $f(S^2_-)$.



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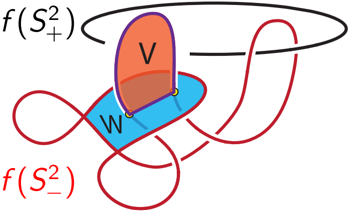

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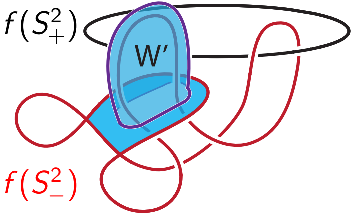

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Theorem (L.)

Let $f : S^2_+ \cup S^2_- \to S^4$ be a link map with $\sigma_-(f) = 0$.

If $\sigma_+(f) = \sum_{p \in \mathrm{self}(f_+)} (t^{n_p} - 1)$,

then $\omega_-(f) = \#\{p : n_p \equiv 2 \mod 4\} \mod 2.$

In particular, there are infinitely many link maps f with $\omega(f) = (0,0)$ but $\sigma(f) \neq (0,0)$.

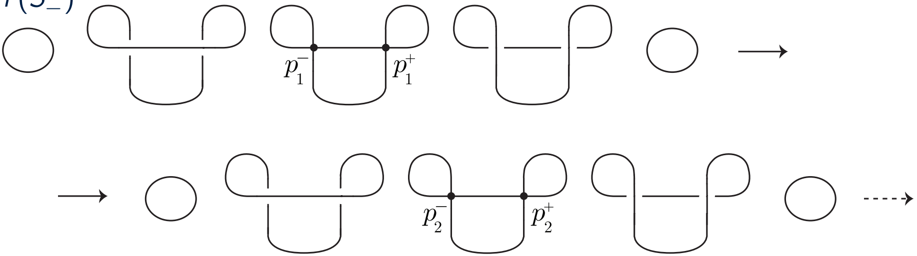
Towards a better invariant?

Let $f : S^2_+ \cup S^2_- \to S^4$ be a link map.

Proposition (S. Kamada)

After a link homotopy, $f(S^2_-)$ is an <u>unknotted immersion</u> in S^4 with $d \geq 0$ pairs of oppositely-signed double points.

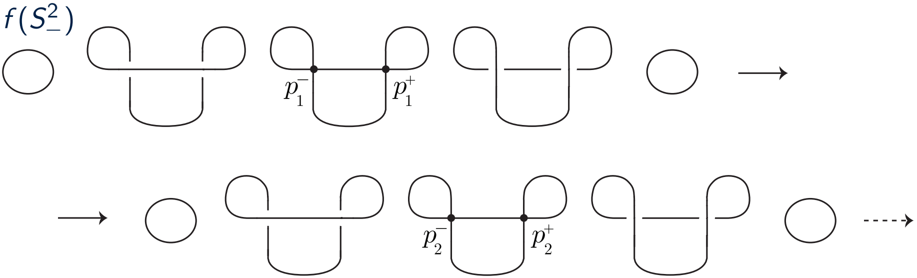
$f(S^2_-)$



Towards a better invariant?

Let $f : S^2_+ \cup S^2_- \to S^4$ be a link map. Write $X_- = S^4 \setminus f(S^2_-)$.

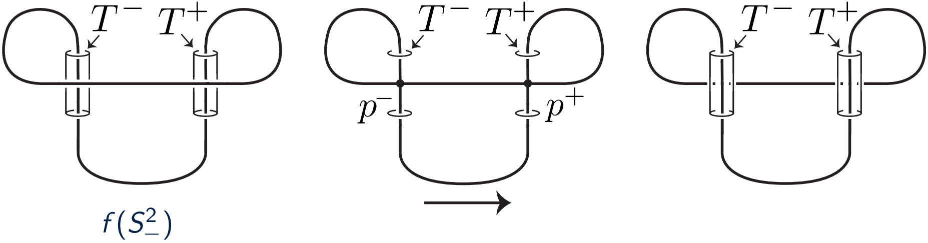
- $\pi_1(X_-) \cong \mathbb{Z}, \qquad \mathbb{Z}\pi_1 = \mathbb{Z}[t, t^{-1}]$
- $\pi_2(X_-) \cong (\bigoplus_{i=1}^{2d} \mathbb{Z})[t, t^{-1}]$



Towards a better invariant?

Construct generators of $\pi_2(X_-) = (\bigoplus_{i=1}^{2d} \mathbb{Z})[t, t^{-1}]$

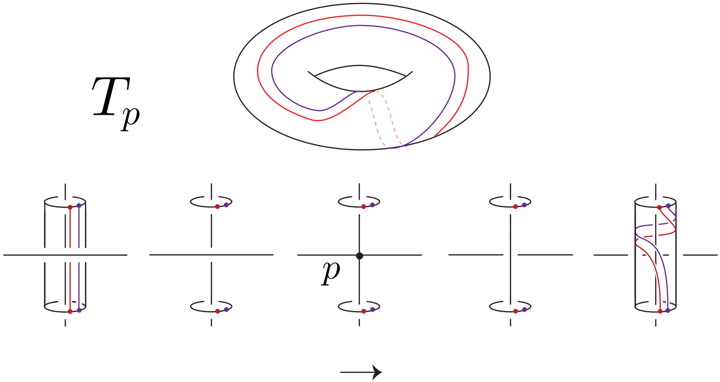
- $H_2(X_-) = \mathbb{Z}^{2d}$
- Generated by linking tori $\{T_i^+, T_i^-\}_{i=1}^d$



Towards a better invariant?

Construct generators of $\pi_2(X_-) = (\bigoplus_{i=1}^{2d} \mathbb{Z})[t, t^{-1}]$.

- Surger T_p to a 2-sphere A_p
- $A_p = (T_p \setminus \text{annulus}) \cup (D_p \cup D'_p)$



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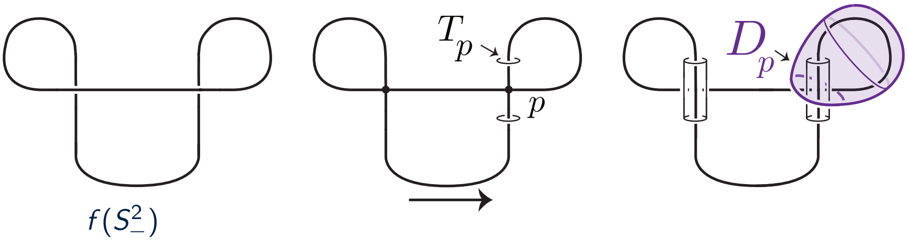

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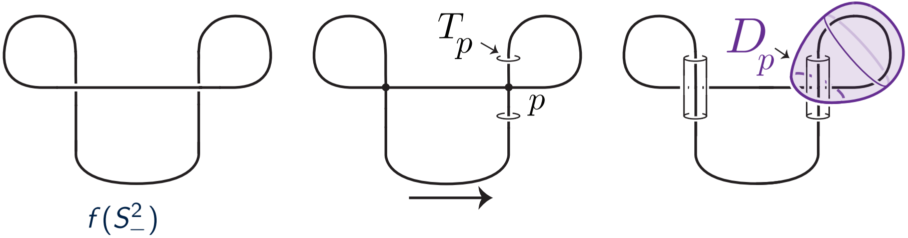
- $A_p = (T_p \setminus \text{annulus}) \cup (D_p \cup D'_p)$
- $\lambda(f_+, A_p) = (1+t)\lambda(f_+, D_p) \in \mathbb{Z}\pi_1(X_-) = \mathbb{Z}[t, t^{-1}]$



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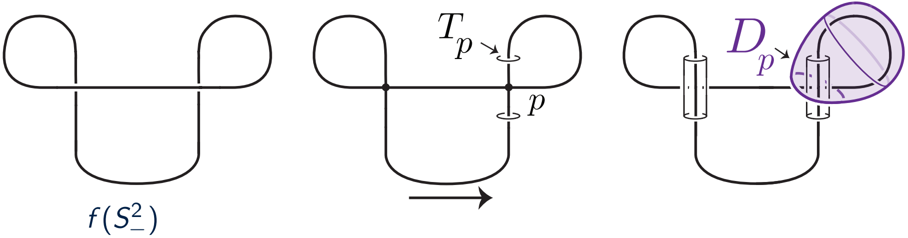

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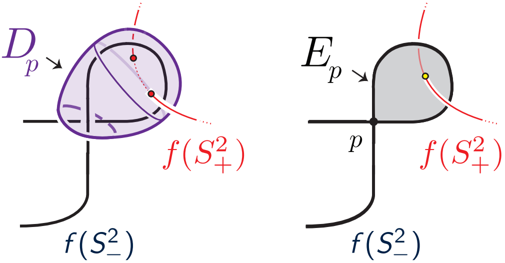

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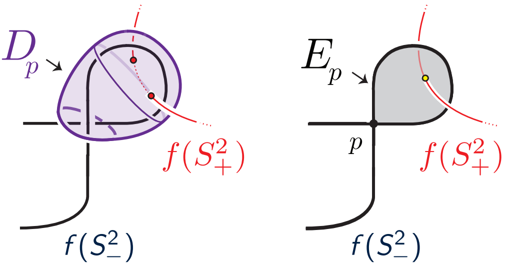

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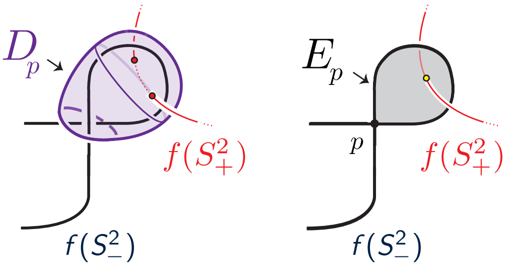
- $\lambda(f_+, E_p) \xrightarrow{t \mapsto 1} n_p$ where $\sigma_-(f) = \sum_p \text{sign}_p(t^{n_p} - 1)$



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After a link homotopy...

- $\pi_2(X_-) = (\bigoplus_{i=1}^{2d} \mathbb{Z})[t, t^{-1}]$ has basis rep. by 2-spheres $\{A_p\}_p$
- $A_p \cap A_q = \varnothing$
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- So: $f_+ \in \pi_2(X_-)$

  $\Rightarrow f_+ = \sum_p c_p(t) A_p, \qquad c_p(1) = n_p$

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Let $f : S^2_+ \cup S^2_- \to S^4$ be a link map with $\sigma_-(f) = 0$.

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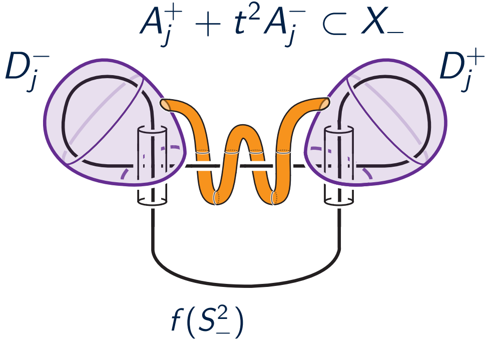

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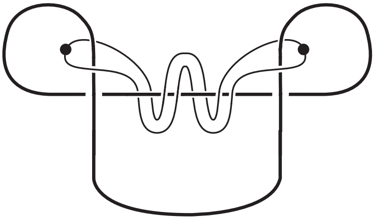

$f(S^2_-)$

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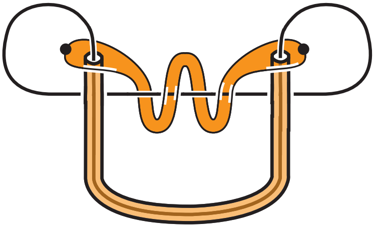

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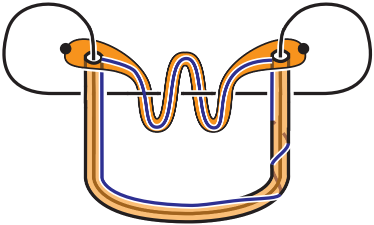

$f(S^2_-)$

Still open

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- **Question:** Can a secondary invariant for 3-component link maps be defined? Is it stronger than σ?