Decomposition of knot complements into right-angled polyhedra

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Motivation: hyperbolic 3-manifolds

Let $\mathbb{H}^3$ denote a 3-dimensional hyperbolic space (Lobachevskii space $\mathbb{L}^3$ in Russia).

Let Γ be a discrete subgroup of $\mathrm{Isom}(\mathbb{H}^3)$ acting without fixed points.

The quotient space $\mathbb{H}^3/\Gamma$ is a hyperbolic 3-manifold.

Klein, 1929, "Non-Euclidean Geometry": Examples of compact hyperbolic 3-manifolds are unknown.

First examples of hyperbolic 3-manifolds of finite volume:

- **Gieseking, 1914:** non-compact, non-orientable.
- **Löbell, 1931:** compact, orientable.
- **Weber, Seifert, 1933:** compact, orientable "dodecahedral hyperbolic space ".

Aim of the talk

We will discuss the construction of hyperbolic 3-manifolds from **right-angled polyhedra**.

- Start with a bounded right-angled polyhedron R in $\mathbb{H}^3$.
  - Which combinatorial polyhedra can be realized as right-angled in $\mathbb{H}^3$?
  - What is a structure of the set of right-angled polyhedra?
- Consider the group G generated by reflections in faces of R.
- Choose a torsion-free subgroup Γ of G.
  - How to find a torsion-free subgroup? Use colourings of a polyhedron!
  - Do different colourings lead to different manifolds?

Outline of the talk

1. The set of all bounded right-angled hyperbolic polyhedra
2. Constructing manifolds from Pogorelov polyhedra
3. The set of all ideal right-angled hyperbolic polyhedra
4. Constructing manifolds from ideal right-angled polyhedra

The set of all bounded right-angled hyperbolic polyhedra

Uniqueness of acute-angled polyhedra in $\mathbb{H}^n$

Let $\mathbb{H}^n$ denote an n-dimensional hyperbolic space.

Andreev, 1970: Any bounded acute-angled (all dihedral angles are at most $\pi/2$) polyhedron in $\mathbb{H}^n$ is uniquely determined by its combinatorial type and dihedral angles.

We will discuss two classes of acute-angled polyhedra:

- Coxeter polyhedra, with dihedral angles of the form π/k, $k \geq 2$.
- Right-angled polyhedra, with all dihedral angles $\pi/2$.

Bounded right-angled polyhedra in $\mathbb{H}^3$

Pogorelov, 1967: A polyhedron P can be realized in $\mathbb{H}^3$ as a bounded right-angled polyhedron if and only if

(1) any vertex is incident to 3 edges (polyhedron is said to be simple);

(2) any face has at least 5 sides;

(3) if a simple closed circuit on the surface of the polyhedron separates two faces (prismatic circuit), then it intersects at least 5 edges;

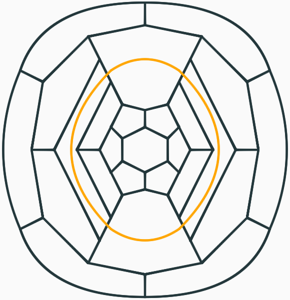
(4) P can be realized in $\mathbb{H}^3$ with dihedral angled less than $\pi/2$.

Andreev, 1970: Condition (4) is not necessary.

Conditions (1) and (3) imply (2).

Conditions (1) and (2) do not imply (3)

The following polyhedron satisfies (1) and (2), but not (3):



There is a closed circuit which separates two 6-gonal faces (top and bottom), but intersects only 4 edges.

Pogorelov polyhedra

Def. A combinatorial polyhedron is Pogorelov polyhedron if

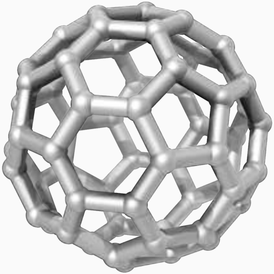
- any vertex is incident to 3 edges (simple polyhedron);
- any prismatic circuit intersects at least 5 edges.

Russian and Ukrainian academician Aleksei Vasil'evich Pogorelov [1919–2002].

A combinatorial polyhedron can be realised as a bounded right-angled polyhedron in $\mathbb{H}^3$ if and only if it is Pogorelov polyhedron.

Fullerenes are Pogorelov polyhedra

If simple polyhedron has only 5- and 6-gonal faces, it is called fullerene.



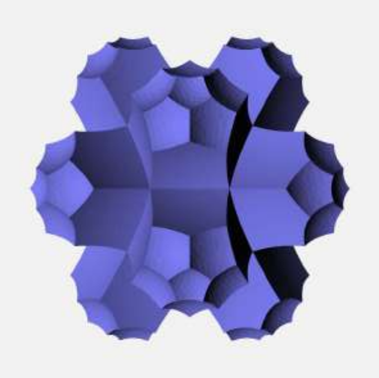
Došlić, 2003; Buchshaber – Erokhovets, 2015: If P is a fullerene, then any prismatic circuit intersects at least 5 edges.

Cor. Fullerenes are Pogorelov polyhedra.

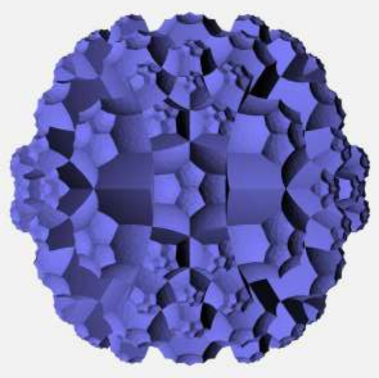
A right-angled dodecahedron in $\mathbb{H}^3$

Combinatorially simplest Pogorelov polyhedron is a dodecahedron.

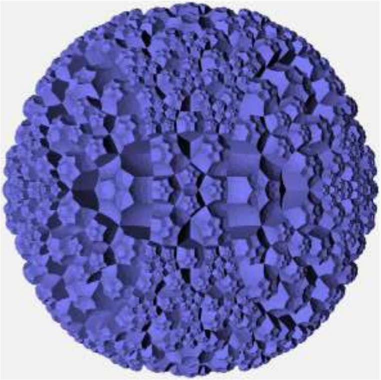
Tiling of $\mathbb{H}^3$ by right-angled dodecahedra, I



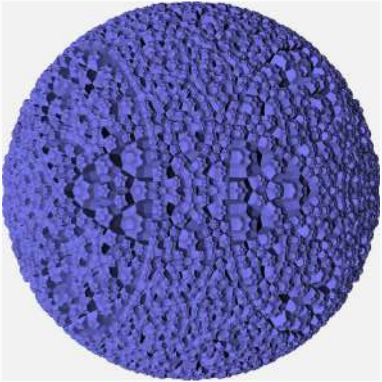
Tiling of $\mathbb{H}^3$ by right-angled dodecahedra, II



Tiling of f $\mathbb{H}^3$ by right-angled dodecahedra, III



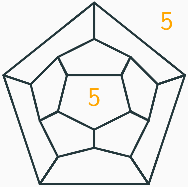
Tiling of f $\mathbb{H}^3$ by right-angled dodecahedra, IV

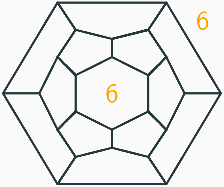


* Images are due to Vladimir Bulatov, www.bulatov.org

An infinite subfamily of the set of Pogorelov polyhedra

V, 1987:: For any integer $n \geqslant 5$ define a right-angled $(2n+2)$-hedron $L(n)$. Polyhedra $L(5)$ and $L(6)$ look as following:





Polyhedra $L(n)$ are said to be Löbelll polyhedra.

German mathematician Frank Richard Löbell [1893–1964].

Two moves for bounded right-angled polyhedra, I

Let $\mathcal{R}$ be the set of all bounded right-angled polyhedra in $\mathbb{H}^3$.

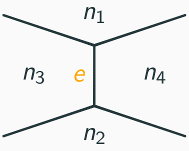
Inoue, 2008: Two moves on $\mathcal{R}$.

- Composition / Decomposition: Consider two combinatorial polyhedra R_1, R_2 with k-gonal faces $F_1 \subset R_1$ and $F_2 \subset R_2$. Then their composition is a union $R = R_1 \cup_{F_1=F_2} R_2$.

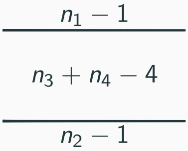
  If $R_1, R_2 \in \mathcal{R}$, then $R \in \mathcal{R}$.

Two moves for bounded right-angled polyhedra, II

- Removing / adding edge: move from R to $R - e$ and inverse:



polyhedron R



polyhedron $R - e$

If $R \in \mathcal{R}$ and e is such that faces F_1 and F_2 have at least 6 sides each and e is not a part of prismatic 5-circuit, then $R - e \in \mathcal{R}$.

Adding edge is known as a Endo–Kroto move for fullerenes. In the case of fullerences $n_1 = n_2 = 6$ and $n_3 = n_4 = 5$.

Reducing to Löbell polyhedra

Inoue, 2008: For any $P_0 \in \mathcal{R}$ there exists a sequence of unions of right-angled hyperbolic polyhedra $P_1, \dots, P_k$ such that:

- each set P_i is obtained from P_{i-1} by decomposition or edge removing,
- any union P_k consists of Löbell polyhedra.

Moreover,

$$\mathrm{vol}(P_0) \geqslant \mathrm{vol}(P_1) \geqslant \mathrm{vol}(P_2) \geqslant \ldots \geqslant \mathrm{vol}(P_k).$$

The set of Pogorelov polyhedra

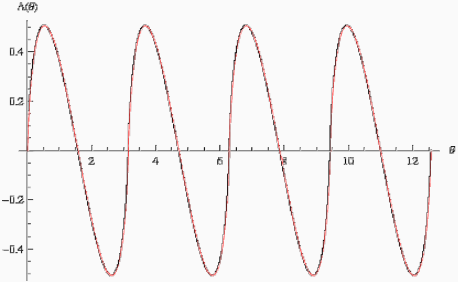
More detailed description:

- Any Löbell polyhedron is non-reducible: it doesn't admit edge removing to another Pogorelov polyhedron or a decomposition into two Pogorelov polyhedra.
- Suppose polyhedron P is Pogorelov, but not Löbell. Then P either can be reduces to another Pogorelov polyhedron by removing an edge, or can be decomposed into two Pogorelov polyhedra, one of which is a dodecahedron.

Lobachevsky function

To express volumes of hyperbolic 3-polyhedra we use the Lobachevsky function

$$\Lambda(\theta) = -\int_0^\theta \log|2\sin(t)|\,\mathrm{d}t.$$



The volume formula for Löbell polyhedra

To each Pogorelov polyhedron R we correspond volume $\mathrm{vol}(R)$ of its right-angled realization in $\mathbb{H}^3$.

V., 1998: Let $L(n)$ denote the Löbell polyhedron, $n \geq 5$. Then

$$\mathrm{vol}(L(n)) = \frac{n}{2}\left[2\Lambda(\theta_n) + \Lambda\left(\theta_n + \frac{\pi}{n}\right) + \Lambda\left(\theta_n - \frac{\pi}{n}\right) + \Lambda\left(\frac{\pi}{2} - 2\theta_n\right)\right],$$

where

$$\theta_n = \frac{\pi}{2} - \arccos\left(\frac{1}{2\cos(\pi/n)}\right).$$

The census of bounded right-angled polyhedra

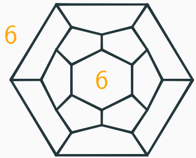
Inoue, 2008: The dodecahedron $L(5)$ and the polyhedron $L(6)$ are the first and the second smallest volume bounded right-angled hyperbolic polyhedra.

Shmel'kov – V., 2011: The eleven smallest volume bounded right-angled hyperbolic polyhedra:

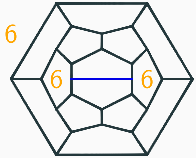
1	4.3062...	$L(5)$	7	8.6124...	$L(5) \cup L(5)$
2	6.0230...	$L(6)$	8	8.6765...	$L(6)_3^3$
3	6.9670...	$L(6)^1$	9	8.8608...	$L(6)_1^3$
4	7.5632...	$L(7)$	10	8.9456...	$L(6)_2^3$
5	7.8699...	$L(6)_1^2$	11	9.0190...	$L(8)$
6	8.0002...	$L(6)_2^2$			

Adding of edges: from $L(6)$ to $L(6)^1$

The polyhedron $L(6)$ and possible faces to add an edge (Endo–Kroto move):



The polyhedron $L(6)^1$ and possible faces to add an edge:



Volume bounds from combinatorics of polyhedra

Atkinson, 2009: Let P be a bounded right-angled hyperbolic polyhedron with F faces. Then

$$\frac{v_8}{16}F - \frac{3v_8}{8} \leqslant \mathrm{vol}(P) < \frac{5v_3}{4}F - \frac{35v_3}{4},$$

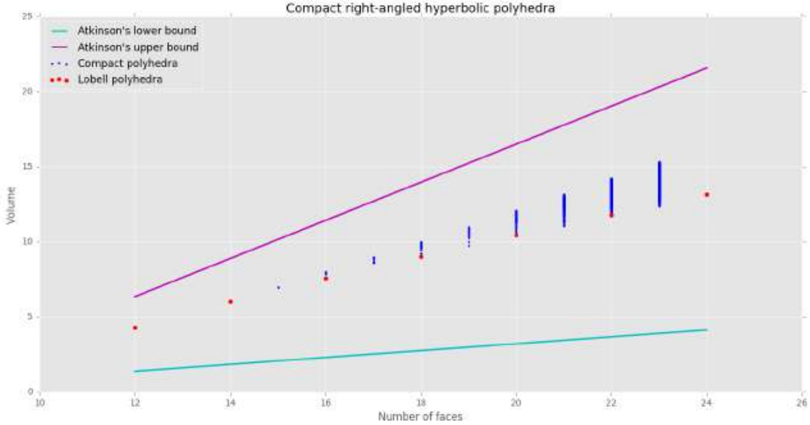
where $v_8 = 3.66386\ldots$ and $v_3 = 1.01494\ldots$.

Matveev – Petronio - V., 2009: For Löbell polyhedron L with F faces we have $\mathrm{vol}(L) \to \frac{5v_3}{8}F - \frac{5v_3}{4}$ as $F \to \infty$.

Inoue, arxiv:1512.0176:
The first 825 bounded right-angled polyhedra are constructed by compositions and edge surgeries. The 825-th smallest right-angled polyhedron has volume $13.4203\ldots$.

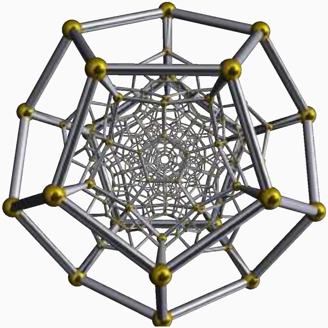
The modern census of bounded right-angled polyhedra

Shmel'kov – V.: about 3.000 smallest bounded right-angled polyhedra.



Bounded right-angled polyhedra in $\mathbb{H}^n$, $n > 3$

There is a bounded right-angled polyhedron in $\mathbb{H}^4$. Combinatorically it is the 120-cell, the convex regular 4-polytope with the boundary composed of 120 dodecahedral cells with 4 meeting at each vertex.



Nikulin 1981: No bounded right-angled polyhedra in $\mathbb{H}^n$ for $n > 4$.

Open problem. Are there bounded right-angled polyhedra in $\mathbb{H}^4$ which are not obtained from the 120-cell?

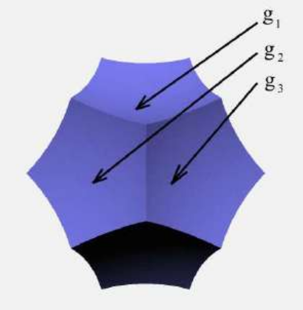
Constructing manifolds from Pogorelov polyhedra

Stabilizer of a vertex

Suppose

- P be a bounded $\pi/2$–polyhedron in $\mathbb{H}^3$;
- G be the group generated by reflections in faces of P.

For each vertex $v \in P$ its stabilizer in G is generated by three reflections g_1, g_2, g_3 and is isomorphic to the eight-element abelian group $(\mathbb{Z}/2\mathbb{Z}) \oplus (\mathbb{Z}/2\mathbb{Z}) \oplus (\mathbb{Z}/2\mathbb{Z}) = \mathbb{Z}_2^3$.



Local linear independence

The group $\mathbb{Z}_2^3$ can be regarded as the finite vector space over the field $GF(2)$ with a basis

$$\{(1,0,0),(0,1,0),(0,0,1)\}.$$

Al-Jubouri, 1980: The kernel Ker φ of an epimorphism $\varphi : G \to \mathbb{Z}_2^3$ is torsion-free if and only if for any vertex v of P images of reflections in faces incident to v are linearly independent in $\mathbb{Z}_2^3$.

The proof was done for a dodecahedron, but can be easy generalized.

Thus, if φ satisfies this local linear independence property then $M = \mathbb{H}^3/\operatorname{Ker}\varphi$ is a closed hyperbolic 3-manifold (orientable or non-orientable) constructed from eight copies of P.

Four colours

Elements $\alpha = (1,0,0)$, $\beta = (0,1,0)$, $\gamma = (0,0,1)$ and $\delta = \alpha + \beta + \gamma = (1,1,1)$ are such that any three of them are linearly independent in $\mathbb{Z}_2^3$.

V., 1987: If $\varphi : G \to \mathbb{Z}_2^3$ is such that for any generator g of G its image $\varphi(g)$ belongs to $\{\alpha, \beta, \gamma, \delta\}$ then $\operatorname{Ker}\varphi$ consists of orientation-preserving isometries.

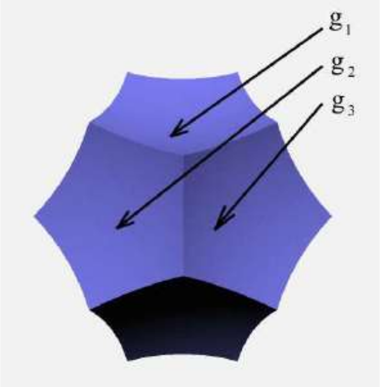
Cor. If an epimorphism $\varphi : G \to \mathbb{Z}_2^3$ is such that

- for any generator g of G its image $\varphi(g)$ belongs to $\{\alpha, \beta, \gamma, \delta\}$;
- for any two adjacent faces their images are different;

then $M = \mathbb{H}^3/\operatorname{Ker}\varphi$ is a closed orientable hyperbolic 3-manifold.

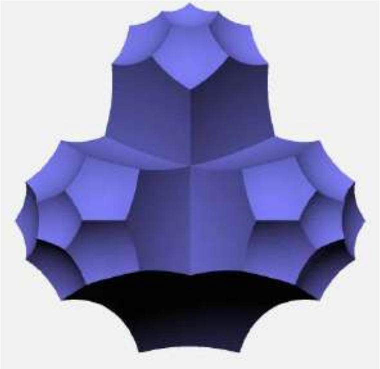
Cor. Any 4-colouring of faces of a Pogorelov polyhedron P determ a closed orientable hyperbolic 3-manifold.

Tiling around a vertex, I



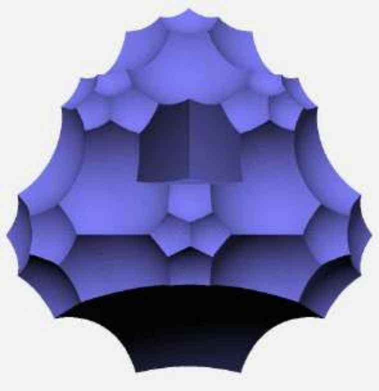
P

Tiling around a vertex, II



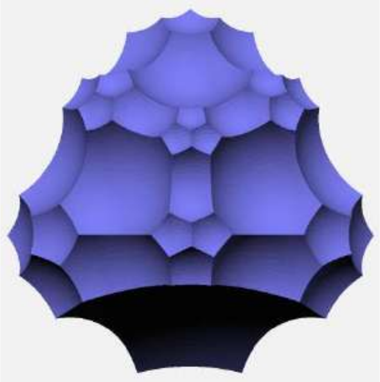
$P \cup g_1(P) \cup g_2(P) \cup g_3(P)$

Tiling around a vertex, III



$P \cup g_1(P) \cup g_2(P) \cup g_3(P) \cup g_1 g_2(P) \cup g_1 g_3(P) \cup g_2 g_3(P)$

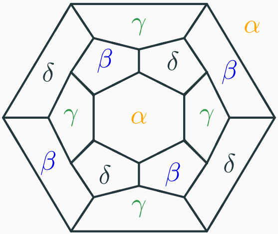
Tiling around a vertex, IV



A fundamental polyhedron for $\operatorname{Ker}\varphi < G$:
$P \cup g_1(P) \cup g_2(P) \cup g_3(P) \cup g_1g_2(P) \cup g_1g_3(P) \cup g_2g_3(P) \cup g_1g_2g_3(P)$

Example: the Löbell manifold

The classical Löbell manifold, the first example of closed orientable hyperbolic 3-manifold in 1931, can be obtained in this way: from the following 4-colouring of $L(6)$:



F. Löbell, Beispiele geschlossene dreidimensionaler Clifford–Kleinischer Räume negative Krümmung, Ber. Verh. Sächs. Akad. Lpz., Math.-Phys. Kl. **83** (1931), 168–174.

When two 4-colourings induce the same manifolds?

Let P be a bounded right-angled hyperbolic polyhedron. Let G be generated by reflections in faces of P, and Σ be the symmetry group of P.

A group G is said to be naturally maximal if $\langle G, \Sigma \rangle$ is maximal discrete group, i.e. is not a proper subgroup of any discrete group of $\mathrm{Isom}(\mathbb{H}^3)$.

V.: Let G be non-arithmetic and naturally maximal. Let $\varphi_1, \varphi_2 : G \to \mathbb{Z}_2^3$ be epimorphisms induced by two 4-colourings. Manifolds $\mathbb{H}^3 / \mathrm{Ker}(\varphi_1)$ and $\mathbb{H}^3 / \mathrm{Ker}(\varphi_2)$ are isometric if and only if 4-coloruings are equivalent.

Example. Let $L(n)$, $n \geq 5$, be the Löbell polyhedron and $G(n)$ be the group generated by reflections in faces of it.

1. Roeder: if $n \neq 5, 6, 8$ then group $G(n)$ is non-arithmetic;
2. Mednykh: if $n \geq 6$ then $G(n)$ is naturally maximal.

Equivalence of 4-colourings

The toric topology approach.

Buchstaber, Erochovets, Masuda, Panov, Park, 2017:
"Cohomological rigidity of manifolds defined by right-angled 3-dimensional polytopes".

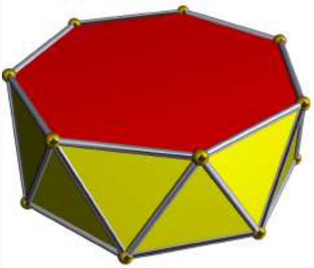
Buchstaber, Panov, 2016:

Let $M = (P, \varphi)$ and $M' = (P', \varphi')$ be hyperbolic 3-manifolds, corresponding to 4-colourings of Pogorelov polyhedra: φ for P and φ' for P'. Then M and M' are diffeomorphic if and only if pairs (P, φ) and (P', φ') are equivalent.

The set of all ideal right-angled hyperbolic polyhedra

Ideal right-angled antiprisms

Let $\mathcal{A}_n$, $n \geqslant 3$, be an ideal (with all vertices at infinity) n-antiprism in $\mathbb{H}^3$ with dihedral angles $\pi/2$. Antiprism $\mathcal{A}_7$ is presented it the figure.



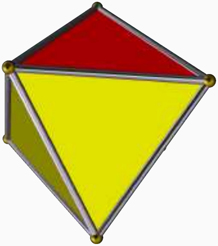
It is known from Thurston's lecture notes (1978) that

$$\mathrm{vol}(\mathcal{A}_n) = 2n \left[\Lambda\left(\frac{\pi}{4} + \frac{\pi}{2n}\right) + \Lambda\left(\frac{\pi}{4} - \frac{\pi}{2n}\right)\right].$$

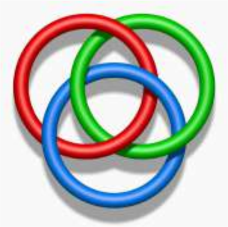
∗ Images are due to Wikipedia, www.wikipedia.org

Ideal right-angled octahedron

Observe that $\mathcal{A}_3$ is an ideal right-angled octahedron.



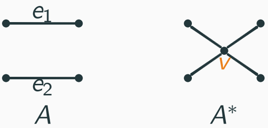
Compare with the diagram of the Borromean rings.



Moves on ideal polyhedra

Let $\mathcal{A}$ be the set of all ideal right-angled polyhedra in $\mathbb{H}^3$. Define a move on the set $\mathcal{A}$.

- Edge twisting: combinatorial transformation from $A \in \mathcal{A}$ to A^*:



Example. An edge-twisting applied to the 4-antiprism.

The set of all ideal right-angled polyhedra

Shmel'kov, 2011 (MSc diploma work, still unpublished):

1. If $A \in \mathcal{A}$ then $A^* \in \mathcal{A}$.
2. The volume increases under an edge twisting move.
3. Every ideal right-angled polyhedron $A \in \mathcal{A}$ can be constructed by a finitely many edge twisting moves from an n-antiprism $\mathcal{A}_n$ for some n.

Ideas of the proof.

1. Rivin, 1992: a polytope $A \in \mathcal{A}$ if and only is 1-skeleton of A is 4-valent and cyclically 6-connected.
2. Schläfli volume variation formula.
3. Brinkmann, Greenberg, Greenhill, McKay, Thomas, Wollan, 2005: generation of simple quadangulations of the sphere.

Census of ideal right-angled polyhedra

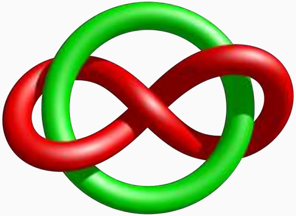
Cor. The octahedron $\mathcal{A}_3$ and polyhedron $\mathcal{A}_4$ are the first and the second smallest volume ideal right-angled polyhedra.

The nineteen smallest volume ideal right-anged hyperbolic polyhedra:

1	3.6638...	$\mathcal{A}_3$	11	10.9915...	$\mathcal{A}_5^{**}(6)$
2	6.0230...	$\mathcal{A}_4$	12	11.1362...	$\mathcal{A}_5^{**}(5)$
3	7.3277...	$\mathcal{A}_4^{*}$	13	11.1362...	$\mathcal{A}_5^{**}(2)$
4	8.1378...	$\mathcal{A}_5$	14	11.4472...	$\mathcal{A}_5^{**}(3)$
5	8.6124...	$\mathcal{A}_4^{**}$	15	11.8017...	$\mathcal{A}_4^{****}(1)$
6	9.6869...	$\mathcal{A}_5^{*}$	16	11.8017...	$\mathcal{A}_6^{*}(1)$
7	10.1494...	$\mathcal{A}_4^{***}$	17	12.0460...	$\mathcal{A}_4^{****}(2)$
8	10.1494...	$\mathcal{A}_6$	18	12.0460...	$\mathcal{A}_6^{*}(2)$
9	10.8060...	$\mathcal{A}_5^{**}(1)$	19	12.1062...	$\mathcal{A}_7$
10	10.9915...	$\mathcal{A}_5^{**}(4)$			

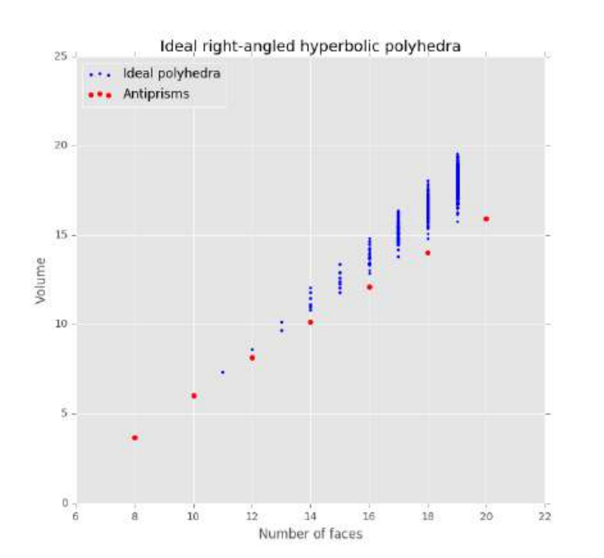
The smallest volume decomposible link

Cor. The Whitehead link complement is the smallest volume link complement that can be decomposed into ideal right-angled polyhedra (one copy of the octahedron $\mathcal{A}(3)$):



The structure of the volume set

Shmel'kov – V.: about 2.000 smallest ideal right-angled polyhedra.



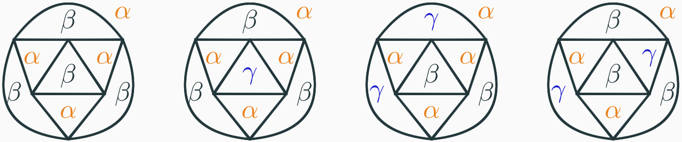
Constructing manifolds from ideal right-angled polyhedra

Construction

Let $A \in \mathcal{A}$ and $G(A)$ be a group, generated by reflections. Let $\varphi : G \to \mathbb{Z}_2^2$ be a surjective homomorphism given by a $\mathbb{Z}_2^2$–colouring of faces of A. Then $M = \mathbb{H}^3/\mathrm{Ker}\,\varphi$ is a cusped hyperbolic 3-manifold.

Moreover, M is orientable if φ corresponds to a 2-colouring (colours $(1,0),(0,1) \in \mathbb{Z}_2^2$) and non-orientable if it corresponds to a 3-colouring (colours $(1,0),(0,1),(1,1) \in \mathbb{Z}_2^2$).

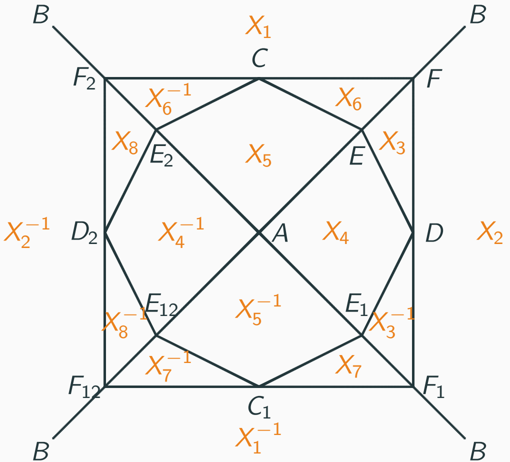
Example. Consider an ideal right-angled octahedron $A(3)$. it is easy to see that $A(3)$ admits one 2-colouring and three 3-colourings as presented in the figure. Denote corresponding epimorphisms by φ_0, φ_1, φ_2, and φ_3.



Kernel of the epimorphism

For the epimorphism $\varphi_0 : G(A(3)) \to \mathbb{Z}_2^2$ denote $\Gamma_0 = \mathrm{Ker}\varphi_0$.

A fundamental polyhedron $\widetilde{A}(3)$ of Γ_0 consists of 4 copies of $A(3)$:



Manifold constraction

$\widetilde{A}$ has 16 faces, 14 ideal vertices

$$A, B, C, C_1, D, D_2, E, E_1, E_2, E_{12}, F, F_1, F_2, F_{12}.$$

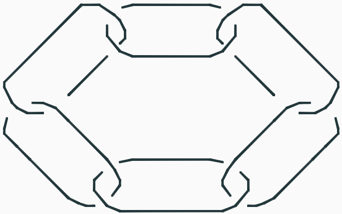
Γ_0 is generated by isometries $x_1, \ldots, x_8$, where $x_i : X_i^{-1} \to X_i$.

Vertices of $\widetilde{A}$ split in 6 classes of equivalent under Γ_0:

$$\{A\}, \{B\}, \{C, C_1\}, \{D, D_2\}, \{E, E_1, E_2, E_{12}\}, \{F, F_1, F_2, F_{12}\}.$$

Each class gives a tori cusp of a manifold $M_0 = \mathbb{H}^3/\Gamma_0$.

M_0 is complement of a 6-chain link. vol $M_0 = 14,65544951\ldots$.



Non-orientable cusped manifolds

We have 3-colourings in non-trivial elements of $\mathbb{Z}_2^2$: φ_1, φ_2, and φ_3.

All of them lead to non-orientable manifolds with 6 cusps.

Cusps of $M_1 = \mathbb{H}^3/\operatorname{Ker}\varphi_1$: 3 tori and 3 Klein bottles.

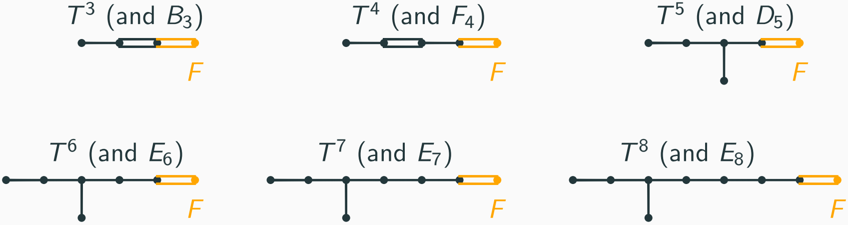
Cusps of $M_2 = \mathbb{H}^3/\operatorname{Ker}\varphi_2$: 2 tori and 4 Klein bottles.

Cusps of $M_3 = \mathbb{H}^3/\operatorname{Ker}\varphi_3$; 6 Klein bottles.

Open problem. Is it true in general case that non-equivalent 3-colourings give non-homeomorphic manifolds?

Finite-volume right-angled polyhedra in $\mathbb{H}^n$, $n \geq 3$

Examples are known for $n \leq 8$ only. Consider simplices $T^3, \ldots, T^8$ given by Coxeter diagrams:



"Black" subdiagrams correspond to finite Coxeter groups: $|B_3| = 2^3 \cdot 3!$, $|F_4| = 1152$, $|D_5| = 2^4 \cdot 5!$, $|E_6| = 72 \cdot 6!$, $|E_7| = 72 \cdot 8!$, $|E_8| = 192 \cdot 10!$.

Dufour, 2010: No finite volume $\pi/2$–polyhedra in $\mathbb{H}^n$ for $n > 12$.

Open problem. What about dimensions $n = 9, 10, 11, 12$?

General construction

Let $P \subset \mathbb{H}^n$ be a right-angled polyhedron and $G(P)$ be a group, generated by reflections in hyperfaces. Denote the set of hyperfaces $\mathcal{F}$. Let homomorphism $\phi : G(P) \to \mathbb{Z}_2^k$, $k \geq n$ be identified with a colouring $\phi : \mathcal{F} \to \mathbb{Z}_2^k$ of hyperfaces. Let colouring $\phi : \mathcal{F} \to \mathbb{Z}_2^k$ be regular, that means

- for any finite vertex of P colours of incident hyperfaces are linear independent as vectors in $\mathbb{Z}_2^k$,
- for any edge of P colours of incident hyperfaces are linear independent.

V. 1987; Davis and Janushkevich, 1991; Garrison and Scott, 2003; Kolpakov, Martelli and Tschantz, 2015; Kolpakov and Slavich, 2016:

Then $\Gamma = \operatorname{Ker} \phi$ is torsion-free and $M = \mathbb{H}^n / \Gamma$ is a hyperbolic manifold.

4-manifold with 1 cusp

Kolpakov, Slavich, 2016:

There are orientable hyperbolic 4-manifolds with 1 cusp.

The manifold $\mathcal{X}$ has unique cusp which is S^1-fibre over a Klein bottle.

The manifold $\mathcal{Y}$ has unique cusp which is a 3-torus.

Open problem. Are there hyperbolic 5-manifolds with 1 cusp?

References

- A. Vesnin, *Right-angled polytopes and hyperbolic 3-manifolds*, Russian Math. Surveys, **72:2** (2017), 147–190.

Thank you!