Homology groups of certain finite quandles

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Basic definitions and notions

Definition

A *quandle* is a nonempty set X together with a binary operation $*: X \times X \to X$ which satisfies the following three properties.

(i) For all $a \in X$, $a * a = a$,

(ii) For all $a, b \in X$, $\exists_1\ c \in X$ s.t $a = c * b$,

(iii) For all $a, b, c \in X$, $(a * b) * c = (a * c) * (b * c)$.

Def. A *rack* is a nonempty set X with a binary operation $*: X \times X \to X$ satisfying (ii), (iii) conditions.

- The function $\sigma_b : X \to X$, defined by $\sigma_b(x) = x * b$ for all $x \in X$, is a permutation of X.

In the view point of a family of permutations, one can restate the definition of a quandle as follows.

Definition

A *quandle* is a nonempty set X with the quandle structure $\Sigma : X \to Map(X, X)$, defined by $x \mapsto \sigma_x$, satisfying the following conditions.

(i) For all $x \in X$, $\sigma_x(x) = x$.

(ii) For all $x \in X$, σ_x is a permutation of X.

(iii) For all $x, y \in X$, $\sigma_x \sigma_y \sigma_x^{-1} = \sigma_{\sigma_x(y)}$

In particular, if $X = \{x_1, x_2, \cdots, x_n\}$ is a finite set, we denote a quandle (X, Σ) by a sequence $[\sigma_1, \sigma_2, \cdots, \sigma_n]$ of permutations σ_i corresponding to x_i.

Example

- Any set X with the binary operation $a * b = a$ for all $a, b \in X$ is a quandle, which is called a *trivial quandle*.
- For the set $X = \{1, 2, \cdots, n\}$, define a binary operation $* : X \times X \to X$ by $i * j = 2j - i \pmod{n}$ for all $i, j \in X$. The pair $(X, *)$ forms a quandle, which is called the *dihedral quandle* of order n and is denoted by R_n.

$*_e$	1	2	3
1	1	1	1
2	2	2	2
3	3	3	3
	[1,	1,	1]

R_3	1	2	3
1	1	3	2
2	3	2	1
3	2	1	3
	[(23),	(13),	(12)]

Definition

Let $(X, *)$ and $(X', *')$ be two quandles.

- A *quandle homomorphism* is a function $f : X \to X'$ if $f(x * y) = f(x) *' f(y)$ for all $x, y \in X$.
- A *quandle isomorphism* is a quandle homomorphism $f : X \to X'$ if it is bijective.
- A *quandle automorphism* of X is a quandle isomorphism from $(X, *)$ onto itself.
- The *automorphism group* $Aut(X)$ of X is the set of all quandle automorphisms of X.
- The *inner automorphism group* $Inn(X)$ of X is the subgroup of $Aut(X)$ generated by $\{\sigma_b, \sigma_b^{-1} | b \in X\}$.

Definition

Let X be a quandle and $x \in X$. From the natural action of $Inn(X)$ on X, one can define

- $Orb(x) = \{ y \in X \mid y = f(x), f \in Inn(X) \}$: the *orbit* of x.
- $Orb(X) = \{ Orb(x) \mid x \in X \}$: the set of all orbits of X.

Under the operation $* : Orb(X) \times Orb(X) \to Orb(X)$ defined by $Orb(x) * Orb(y) = Orb(x * y)$, $Orb(X)$ is the *trivial* quandle, which is called the *orbit quandle* of X.

Rack and Quandle homology groups of $(X, *)$

Definition

- $(X, *)$: a quandle (or a rack)
- $C_n^R(X) =$ the free abelian group generated by $\{(x_1, x_2, \cdots, x_n) | x_i \in X\}$.
- $\partial_n : C_n^R(X) \to C_{n-1}^R(X)$: the homomorphism defined by

$$\partial_n(x_1, x_2, \cdots, x_n) = \sum_{i=2}^{n} (-1)^i [(x_1, x_2, \cdots, x_{i-1}, x_{i+1}, \cdots, x_n)$$

$$-(x_1 * x_i, x_2 * x_i, \cdots, x_{i-1} * x_i, x_{i+1}, \cdots, x_n)]$$

  for $n \geqslant 2$ and $\partial_n = 0$ for $n \leqslant 1$.
- $C_*^R(X) = \{C_n^R(X), \partial_n\}$: a chain complex

Rack and Quandle homology groups of $(X, *)$

Definition

- $C_n^D(X) = \begin{cases} <\{(x_1, x_2, \cdots, x_n) | x_i = x_{i+1}\}>, & n \geqslant 2, \\ 0, & \text{otherwise} \end{cases}$
- $C_*^D(X) = \{C_n^D(X), \partial_n\}$: a sub-complex of $C_*^R(X)$
- $C_n^Q(X) = C_n^R(X)/C_n^D(X)$
- $C_*^Q(X) = \{C_n^Q(X), \partial'_n\}$: a chain complex

For an abelian group G, one can obtain the homology groups

- $H_n^R(X; G)$: the *nth rack homology group*
- $H_n^D(X; G)$: the *nth degeneration homology group*
- $H_n^Q(X; G)$: the *nth quandle homology group*

History

Prop. (Carter-Jelsovsky-Kamada-Saito,2001)
Let X be a quandle. Then there is a long exact sequence

$$\cdots \xrightarrow{\partial_*} H_n^D(X;G) \xrightarrow{i_*} H_n^R(X;G) \xrightarrow{j_*} H_n^Q(X;G) \xrightarrow{\partial_*} H_{n-1}^D(X;G) \to \cdots .$$

which is natural with respect to homomorphisms induced from quandle homomorphisms.

Prop. (Carter-Jelsovsky-Kamada-Saito,2001)
Let X be a quandle with $|Orb(X)| = m$. Then

(1) $H_1^D(X) = 0$,
(2) $H_1^R(X) = H_1^Q(X) = \mathbb{Z}^m$,
(3) $H_2^D(X) = \mathbb{Z}^m$.

Prop. (Carter-Jelsovsky-Kamada-Saito, Litherland-Nelson,2003)
The boundary operators $\partial_* : H_n^Q(X) \to H_{n-1}^D(X)$ are the 0-maps, so that the sequence is decomposed into short exact sequences

$$0 \to H_n^D(X) \to H_n^R(X) \to H_n^Q(X) \to 0.$$

Free part

Note that $H_n^W(X), (W = R, D, Q)$, is a finitely generated abelian group. The free part of $H_n^W(X), (W = R, D, Q)$ is known as follows:

Prop. (Litherland-Nelson, Etingof-Grana,2003)
Let X be a finite rack and $Orb(X)$ the orbit quandle of X.

- $rank(H_n^D(X)) = rank(H_n^W(Orb(X)))$,
- $rank(H_n^R(X)) = rank(H_n^W(Orb(X)))$,
- $rank(H_n^Q(X)) = rank(H_n^W(Orb(X)))$.

Rmk. For the trivial quandle T_m of order m,

- $rank(H_n^D(T_m)) = m^n - m(m-1)^{n-1}$,
- $rank(H_n^R(T_m)) = m^n$,
- $rank(H_n^Q(T_m)) = m(m-1)^{n-1}$.

Torsion part

Prop. (Litherland-Nelson, 2003)
For a finite rack X with homogeneous orbits, the torsion subgroup of $H_n^W(X)$ is annihilated by $|X|^n$ where $W = R, D$ and Q.

Prop. (Etingof-Grana, 2003)
For a finite rack X, the only primes which can appear in the torsion of $H_n^R(X)$ are those dividing $|Inn(X)|$.

Prop. (Niebrzydowski-Przytycki, 2009)
For any $n > 1$, $torH_n^R(R_3)$ is annihilated by 3.

Conj. (Niebrzydowski-Przytycki,2009)
For an odd p, $torH_n^Q(R_p) = \mathbb{Z}_p^{f_n}$, where $f_n = f_{n-1} + f_{n-3}$ and $f(1) = f(2) = 0, f(3) = 1$.

Prop. (Nosaka, Przytycki-Yang,2015)
For a finite quasigroup quandle X, $torH_n^R(X)$ is annihilated by $|X|$.

Consider the following operation table

$*$	1	2	3	4	5	6	7	8
1	1	1	2	2	1	1	1	1
2	2	2	1	1	2	2	2	2
3	4	4	3	3	3	3	3	3
4	3	3	4	4	4	4	4	4
5	5	5	5	5	5	8	6	7
6	6	6	6	6	7	6	8	5
7	7	7	7	7	8	5	7	6
8	8	8	8	8	6	7	5	8

Notice that the sub-tables in diagonal are the operation table for two quandles R_4 and C_4, while the sub-tables in off-diagonal are trivial.

R_4	1	2	3	4
1	1	1	2	2
2	2	2	1	1
3	4	4	3	3
4	3	3	4	4

C_4	1	2	3	4
1	1	4	2	3
2	3	2	4	1
3	4	1	3	2
4	2	3	1	4

Recall that for any quandle X, $H_1^R(X) = \mathbb{Z}^m$ where $m = |Orb(X)|$.

Lemma

For two finite quandles $(Q_1, *_1)$ and $(Q_2, *_2)$, let $(X, *)$ be a quandle for which table is as follows.

Q_1	id
id	Q_2

Then

$$torH_2^R(X) = torH_2^R(Q_1) \oplus torH_2^R(Q_2)$$

Indeed, $H_2^R(X) = H_2^R(Q_1) \oplus H_2^R(Q_2) \oplus \mathbb{Z}^{2|Orb(Q_1)||Orb(Q_2)|}$.

Sketch of proof. Let $X = \{1, \cdots, n, n+1, \cdots, n+m\}$.

The free abelian group $C_2^R(X)$ is generated by

$$\boxed{\begin{array}{cccccc} (1,1) & \cdots & (1,n) & (1,n+1) & \cdots & (1,n+m) \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ (n,1) & \cdots & (n,n) & (n,n+1) & \cdots & (n,n+m) \\ (n+1,1) & \cdots & (n+1,n) & (n+1,n+1) & \cdots & (n+1,i_2-1) \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ (n+m,1) & \cdots & (n+m,n) & (n+m,n+1) & \cdots & (n+m,n+m) \end{array}}$$

Consider the image of ∂_2 and ∂_3, respectively.

$$\partial_2(x,y) = (x) - (x*y) = \begin{cases} (x) - (x *_1 y), & 1 \leqslant x \leqslant n, 1 \leqslant y \leqslant n; \\ 0, & 1 \leqslant x \leqslant n, n+1 \leqslant y \leqslant n+m; \\ 0, & n+1 \leqslant x \leqslant n+m, 1 \leqslant y \leqslant n; \\ (x) - (x *_2 y), & n+1 \leqslant x \leqslant n+m, n+1 \leqslant y \leqslant n+m, \end{cases}$$

$$\partial_3(x,y,z) = (x,z) - (x*y,z) - (x,y) + (x*z, y*z)$$

$$= \begin{cases} (x,z) - (x *_1 y, z) - (x,y) + (x *_1 z, y *_1 z), & 1 \leqslant x \leqslant n, 1 \leqslant y \leqslant n, 1 \leqslant z \leqslant n; \\ (x,z) - (x *_1 y, z), & 1 \leqslant x \leqslant n, 1 \leqslant y \leqslant n, n+1 \leqslant z \leqslant n+m; \\ -(x,y) + (x *_1 z, y), & 1 \leqslant x \leqslant n, n+1 \leqslant y \leqslant n+m, 1 \leqslant z \leqslant n; \\ -(x,y) + (x, y *_2 z), & 1 \leqslant x \leqslant n, n+1 \leqslant y \leqslant n+m, n+1 \leqslant z \leqslant n+m; \\ -(x,y) + (x, y *_1 z), & n+1 \leqslant x \leqslant n+m, 1 \leqslant y \leqslant n, 1 \leqslant z \leqslant n; \\ -(x,y) + (z *_2 z, y), & n+1 \leqslant x \leqslant n+m, 1 \leqslant y \leqslant n, n+1 \leqslant z \leqslant n+m; \\ (x,z) - (x *_2 y, z), & n+1 \leqslant x \leqslant n+m, n+1 \leqslant y \leqslant n+m, 1 \leqslant z \leqslant n; \\ (x,z) - (x *_2 y, z) - (x,y) + (x *_2 z, y *_2 z), & n+1 \leqslant x \leqslant n+m, n+1 \leqslant y \leqslant n+m, n+1 \leqslant z \leqslant n+m. \end{cases}$$

$(1,1)$	$\cdots$	$(1,n)$	$(1,n+1)$	$\cdots$	$(1,n+m)$
$\vdots$	$\ddots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$
$(n,1)$	$\cdots$	(n,n)	$(n,n+1)$	$\cdots$	$(n,n+m)$
$(n+1,1)$	$\cdots$	$(n+1,n)$	$(n+1,n+1)$	$\cdots$	$(n+1,n+m)$
$\vdots$	$\ddots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$
$(n+m,1)$	$\cdots$	$(n+m,n)$	$(n+m,n+1)$	$\cdots$	$(n+m,n+m)$

$$
\begin{array}{|ccc|ccc|}
\hline
 & & & (1, n+1) & \cdots & (1, n+m) \\
 & H_2^R(Q_1) & & \vdots & \ddots & \vdots \\
 & & & (n, n+1) & \cdots & (n, n+m) \\
\hline
(n+1, 1) & \cdots & (n+1, n) & & & \\
\vdots & \ddots & \vdots & & H_2^R(Q_2) & \\
(n+m, 1) & \cdots & (n+m, n) & & & \\
\hline
\end{array}
$$

$H_2^R(Q_1)$	$\lvert Orb(Q_1)\rvert \times \lvert Orb(Q_2)\rvert$
$\lvert Orb(Q_2)\rvert \times \lvert Orb(Q_1)\rvert$	$H_2^R(Q_2)$

$$\therefore torH_2^R(X) = torH_2^R(Q_1) \oplus torH_2^R(Q_2).$$

(In fact, $H_2^R(X) = H_2^R(Q_1) \oplus H_2^R(Q_2) \oplus \mathbb{Z}^{2|Orb(Q_1)||Orb(Q_2)|}$.)

□

Example

Let $(X, *)$ be a quandle which is a disjoint union of two quandles of order 4.

R_4	1	2	3	4
1	1	1	2	2
2	2	2	1	1
3	4	4	3	3
4	3	3	4	4

C_4	1	2	3	4
1	1	4	2	3
2	3	2	4	1
3	4	1	3	2
4	2	3	1	4

$\Rightarrow$

$*$	1	2	3	4	5	6	7	8
1	1	1	2	2	1	1	1	1
2	2	2	1	1	2	2	2	2
3	4	4	3	3	3	3	3	3
4	3	3	4	4	4	4	4	4
5	5	5	5	5	5	8	6	7
6	6	6	6	6	7	6	8	5
7	7	7	7	7	8	5	7	6
8	8	8	8	8	6	7	5	8

It is known that $torH_2^R(R_4) = \mathbb{Z}_2^2$ and $torH_2^R(C_4) = \mathbb{Z}_2$. Hence we have $torH_2^R(X) = \mathbb{Z}_2^3$. In fact,

$$H_2^R(X) = (\mathbb{Z}^4 \oplus \mathbb{Z}_2^2) \oplus (\mathbb{Z} \oplus \mathbb{Z}_2) \oplus \mathbb{Z}^{2\times 2\times 1} = \mathbb{Z}^9 \oplus \mathbb{Z}_2^3.$$

Now, consider the following two quandle tables, in which two diagonals and the lower off-diagonal are trivial.

$*$	1	2	3	4	5	6	7
1	1	1	1	1	1	1	2
2	2	2	2	2	2	2	3
3	3	3	3	3	3	3	1
4	4	4	4	4	4	4	5
5	5	5	5	5	5	5	4
6	6	6	6	6	6	6	6
7	7	7	7	7	7	7	7

$*$	1	2	3	4	5	6	7
1	1	1	1	1	1	2	3
2	2	2	2	2	2	3	1
3	3	3	3	3	3	1	2
4	4	4	4	4	4	5	4
5	5	5	5	5	5	4	5
6	6	6	6	6	6	6	6
7	7	7	7	7	7	7	7

Proposition (B. & Kim)

Let $X = \{1, \cdots, k, k+1, \cdots, k+m\}$ *and* $\tau_{k+1}, \cdots, \tau_{k+m} \in S_k$. *Consider the operation table in the following form;*

$$\begin{array}{|c|c|}\hline id & R \\ \hline id & id \\ \hline \end{array} = \begin{array}{|c|c|}\hline 1 \cdots 1 & \tau_{k+1} \cdots \tau_{k+m} \\ \hline 1 \cdots 1 & 1 \cdots 1 \\ \hline \end{array}$$

Then X *is a quandle under the operation if and only if* $\tau_i \tau_j = \tau_j \tau_i$, *for all* i, j.

Theorem (B. & Choi)

Let $\tau \in S_{n-1}$ *and let* $\sigma = (\tau)(n) \in S_n$*. Let* $X = \{1, 2, \cdots, n\}$ *be the quandle whose operation table is given as follows.*

$$\begin{array}{|c|c|} \hline 1 \cdots 1 & \tau \\ \hline n \cdots n & n \\ \hline \end{array} = \begin{array}{|c|c|} \hline 1 \cdots 1 & \sigma \\ \hline \end{array}$$

Then $H_2^R(X)$ *and* $H_3^R(X)$ *are torsion-free, and hence*

$$H_2^R(X) = \mathbb{Z}^{k^2} \text{ and } H_3^R(Q) = \mathbb{Z}^{k^3}$$

where k *is the number of disjoint cycles of* σ*.*

Sketch of proof for $H_3^R(X)$**.**
Suppose that
$\sigma = (1\ 2 \cdots i_1)(i_1 + 1\ \ i_1 + 2 \cdots i_2) \cdots (i_{k-2} + 1\ \ i_{k-2} + 2 \cdots n - 1)(n)$.

$Im\partial_4$

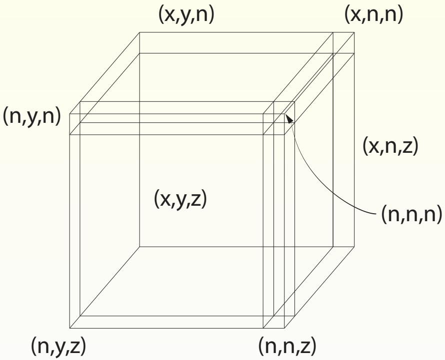
$\partial_4(x,y,z,w) = (x,z,w) - (x*y,z,w) - (x,y,w) + (x*z,y*z,w) + (x,y,z) - (x*w,y*w,z*w)$

1. $\partial_4(x,y,z,w) = 0$
2. $\partial_4(x,y,z,n) = (x,y,z) - (x*n,y*n,z*n)$
3. $\partial_4(x,y,n,w) = -(x,y,w) + (x*n,y*n,w)$
4. $\partial_4(x,y,n,n) = 0$
5. $\partial_4(x,n,z,w) = (x,z,w) - (x*n,z,w)$
6. $\partial_4(x,n,z,n) = (x,z,n) - (x*n,z,n) + (x,n,z) - (x*n,n*n,z*n)$
7. $\partial_4(x,n,n,w) = 0$
8. $\partial_4(x,n,n,n) = (x,n,n) - (x*n,n,n)$
9. $\partial_4(n,y,z,w) = 0$
10. $\partial_4(n,y,z,n) = (n,y,z) - (n*n,y*n,z*n)$
11. $\partial_4(n,y,n,w) = -(n,y,w) + (n*n,y*n,w)$
12. $\partial_4(n,y,n,n) = 0$
13. $\partial_4(n,n,z,w) = 0$
14. $\partial_4(n,n,z,n) = (n,n,z) - (n*n,n*n,z*n)$
15. $\partial_4(n,n,n,w) = 0$
16. $\partial_4(n,n,n,n) = 0$

For every $x, y, z, w = 1, 2, \cdots, n-1$,

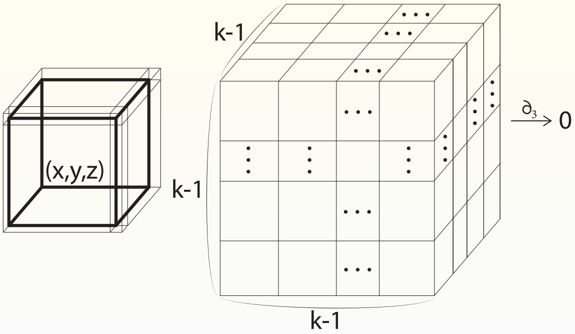
$$
\begin{cases}
\partial_4(x,y,z,n) = & (x,y,z) - (\sigma(x),\sigma(y),\sigma(z)) \\
\partial_4(x,y,n,w) = & -(x,y,w) + (\sigma(x),\sigma(y),w) \\
\partial_4(x,n,z,w) = & (x,z,w) - (\sigma(x),z,w) \\
 & \\
\partial_4(x,n,z,n) = & (x,z,n) - (\sigma(x),z,n) + (x,n,z) - (\sigma(x),n,\sigma(z)) \\
 & \\
\partial_4(x,n,n,n) = & (x,n,n) - (\sigma(x),n,n) \\
 & \\
\partial_4(n,y,z,n) = & (n,y,z) - (n,\sigma(y),\sigma(z)) \\
\partial_4(n,y,n,w) = & -(n,y,w) + (n,\sigma(y),w) \\
 & \\
\partial_4(n,n,z,n) = & (n,n,z) - (n,n,\sigma(z))
\end{cases}
$$

The values of $Im\partial_4$ and $Ker\partial_3$ are depend to the cycles of σ.



Case 1. (x, y, z)

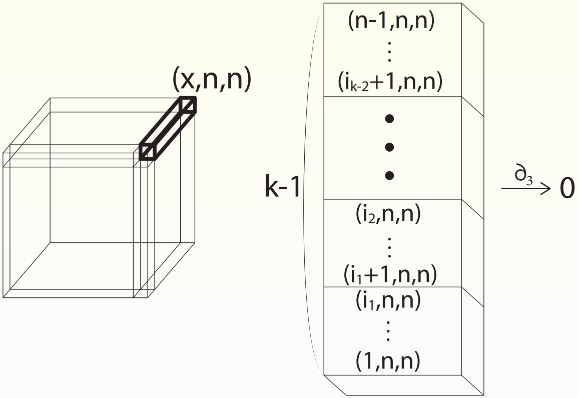
- $Im\partial_4$: $(x, y, z) = (\sigma(x), \sigma(y), \sigma(z)) = (\sigma(x), \sigma(y), z) = (\sigma(x), y, z)$
- $Ker\partial_3$: $\partial_3(x, y, z) = 0$



$\therefore$ (k-1)^3 generators

Case 2. (x, n, n)

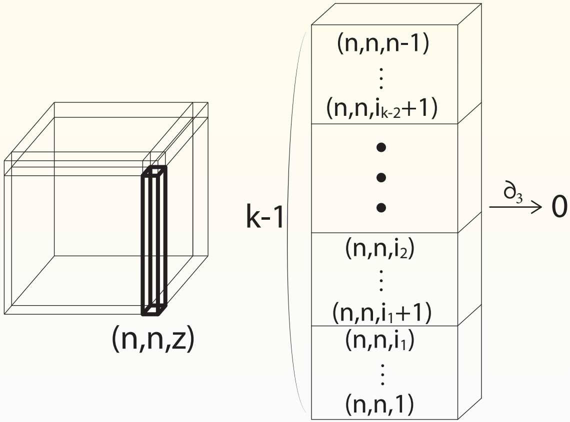
- $Im\partial_4 : (x, n, n) = (\sigma(x), n, n)$
- $Ker\partial_3 : \partial_3(x, n, n) = 0$



$\therefore$ (k-1) generators

Case 3. (n, n, z)

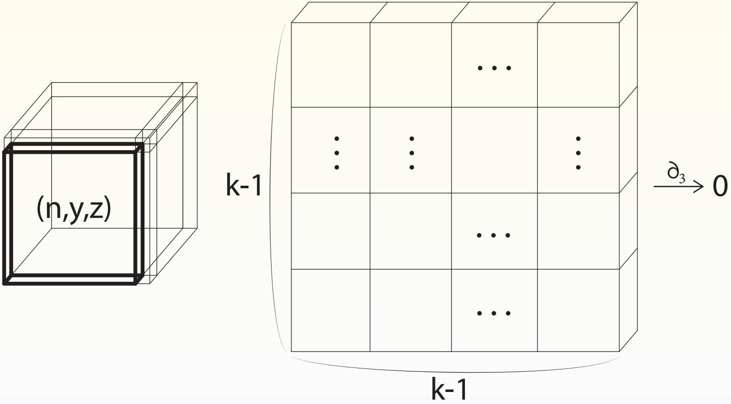
- $Im\partial_4 : (n, n, z) = (n, n, \sigma(z))$
- $Ker\partial_3 : \partial_3(n, n, z) = 0$



$\therefore$ (k-1) generators

Case 4. (n, y, z)

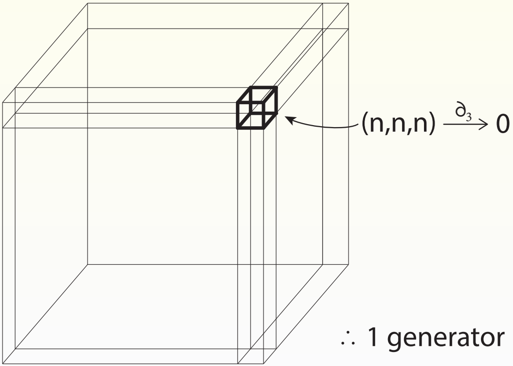
- $Im\partial_4 : (n, y, z) = (n, \sigma(y), \sigma(z)) = (n, \sigma(y), z)$
- $Ker\partial_3 : \partial_3(n, y, z) = 0$



$\therefore$ $(k\text{-}1)^2$ generators

Case 5. (n, n, n)

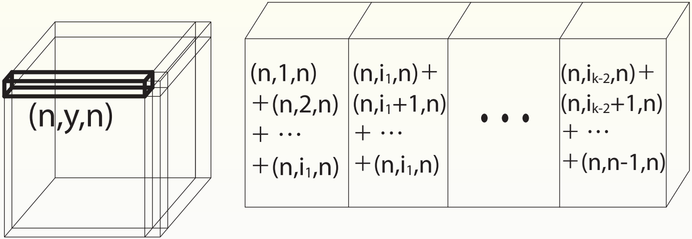
- $Im\partial_4$: No relation
- $Ker\partial_3$: $\partial_3(n, n, n) = 0$



$\therefore$ 1 generator

Case 6. (n, y, n)

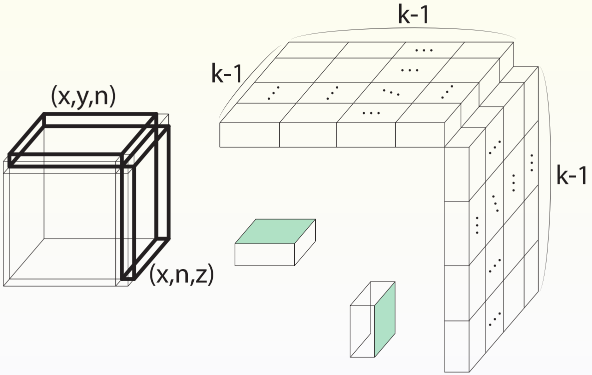
- $Im\partial_4$: No relation
- $Ker\partial_3$: $\partial_3(n, y, n) = -(n, y) + (n, \sigma(y))$
  $\Rightarrow$ Solve $\partial_3\left(\sum_{y=1}^{n-1} \beta_y(n, y, n)\right) = 0.$



$\therefore$ (k-1) generators

Case 7. $(x, z, n), (x, n, z)$

- $Im\partial_4$: $(x, z, n) - (\sigma(x), z, n) + (x, n, z) - (\sigma(x), n, \sigma(z)) = 0$
- $Ker\partial_3$: $\partial_3(x, z, n) = -(x, z) + (\sigma(x), \sigma(z))$
  $\partial_3(x, n, z) = (x, z) - (\sigma(x), z)$



$Im\partial_4$ of Case 7

- Rows : $r_{1,1}, r_{1,2}, \cdots, r_{l_t,l_s}$
  Columns : $(x,n,z),(x,z,n)$
  (oder:$(l_t,l_s,n),(l_t,l_s-1,n),\cdots,(1,1,n) || (l_t,n,l_s),(l_t,n,l_s-1),\cdots,(1,n,1)$)

$$\left(\begin{array}{cccccc|cccccc} I & & & & & -I & I & & & & & -I_s \\ -I & I & & & & & -I_s & I & & & & \\ & -I & I & & & & & -I_s & I & & & \\ & & \ddots & \ddots & & & & & \ddots & \ddots & & \\ & & & & I & & & & & & I & \\ & & & & -I & I & & & & & -I_s & I \end{array}\right)$$

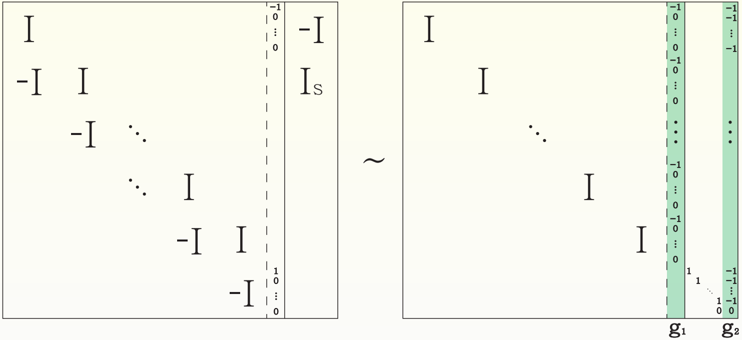
where $I = \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{pmatrix}$ and $I_s = \begin{pmatrix} 0 & 0 & \cdots & 0 & 1 \\ 1 & 0 & \cdots & 0 & 0 \\ \vdots & \ddots & \ddots & \vdots & \vdots \\ 0 & 0 & \ddots & 0 & 0 \\ 0 & 0 & \cdots & 1 & 0 \end{pmatrix}$.

$Im\partial_4$ of Case 7

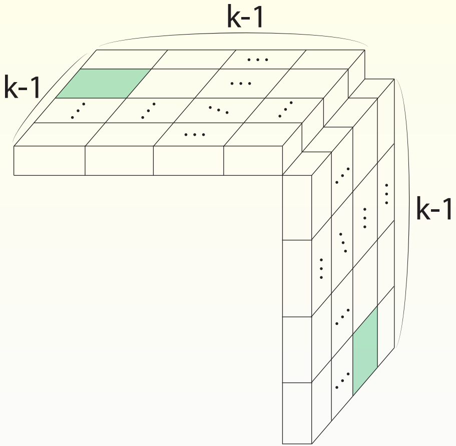
$$\left[\begin{array}{ccccc|cccccc} I & & & & -I & I & & & & & -I_s \\ & I & & & -I & I-I_s & I & & & & -I_s \\ & & I & & -I & I-I_s & I-I_s & I & & & -I_s \\ & & & \ddots & \vdots & \vdots & \vdots & \ddots & \ddots & \vdots & \vdots \\ & & & I & -I & I-I_s & I-I_s & I-I_s & I-I_s & I & -I_s \\ & & & & & U & U & U & U & U & U \end{array}\right]$$

where $I-I_s = \left[\begin{array}{ccccc} 1 & 0 & \cdots & 0 & -1 \\ -1 & 1 & \cdots & 0 & 0 \\ \vdots & \ddots & \ddots & \vdots & \vdots \\ 0 & 0 & \ddots & 1 & 0 \\ 0 & 0 & \cdots & -1 & 1 \end{array}\right]$ and $U = \left[\begin{array}{ccccc} 1 & 0 & \cdots & 0 & -1 \\ 0 & 1 & \cdots & 0 & -1 \\ \vdots & \ddots & \ddots & \vdots & \vdots \\ 0 & 0 & \ddots & 1 & -1 \\ 0 & 0 & \cdots & 0 & 0 \end{array}\right]$

$Ker\partial_3$ of Case 7

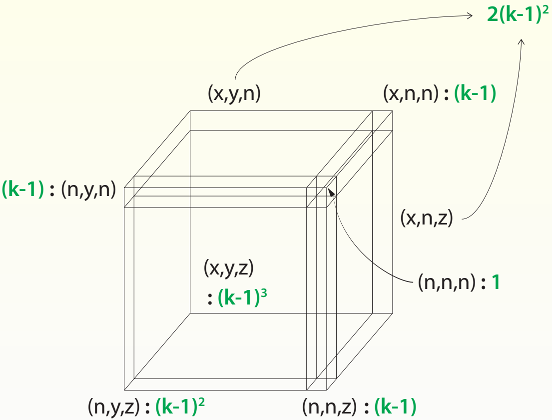


Case 7. $(x, z, n), (x, n, z)$



$\therefore$ $2(k\text{-}1)^2$ generators

$H_3^R(X)$



$\therefore$ k^3 generators & No relations

Example

Let $(X, *)$ be a quandle as follows.

$*$	1	2	3	4	5	6	7
1	1	1	1	1	1	1	2
2	2	2	2	2	2	2	3
3	3	3	3	3	3	3	1
4	4	4	4	4	4	4	5
5	5	5	5	5	5	5	4
6	6	6	6	6	6	6	6
7	7	7	7	7	7	7	7

By the above theorem, one can see that $H_2^R(X)$ and $H_3^R(X)$ are torsion-free, and hence

$$H_2^R(X) = \mathbb{Z}^{4^2} = \mathbb{Z}^{16} \text{ and } H_3^R(X) = \mathbb{Z}^{4^3} = \mathbb{Z}^{64}.$$

Theorem (B. & Choi)

Let $\tau \in S_{n-2}$ and let $\sigma = (\tau)(n-1)(n) \in S_n$. Let $X = \{1, 2, \cdots, n\}$ be the quandle whose operation table is given as follows.

$$\begin{array}{|c|c|} \hline 1 \cdots 1 & \tau \ \tau^i \\ \hline 1 \cdots 1 & 1 \quad 1 \\ \hline \end{array} = \begin{array}{|c|c|} \hline 1 \cdots 1 & \sigma \ \sigma^i \\ \hline \end{array}$$

Then $H_2^R(X)$ is torsion-free.
(Indeed, $H_2^R(Q) = \mathbb{Z}^{k^2}$, where k is the number of disjoint cycles of σ.)

Sketch of proof.

- Put $\sigma = \tau(n-1)(n) \in S_n$.
- $\sigma = (1\ 2 \cdots i_1)(i_1{+}1\ i_1{+}2 \cdots i_2) \cdots (i_{k-3}{+}1\ i_{k-3}{+}2 \cdots n{-}2)(n{-}1)(n)$.

$C_2^R(X) =$

$$\left\{ \begin{array}{ccccccc} (1,1) & (1,2) & (1,3) & \cdots & (1,n-2) & (1,n-1) & (1,n) \\ (2,1) & (2,2) & (2,3) & \cdots & (2,n-2) & (2,n-1) & (2,n) \\ (3,1) & (3,2) & (3,3) & \cdots & (3,n-2) & (3,n-1) & (3,n) \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ (n-2,1) & (n-2,2) & (n-2,3) & \cdots & (n-2,n-2) & (n-2,n-1) & (n-2,n) \\ (n-1,1) & (n-1,2) & (n-1,3) & \cdots & (n-1,n-2) & (n-1,n-1) & (n-1,n) \\ (n,1) & (n,2) & (n,3) & \cdots & (n,n-2) & (n,n-1) & (n,n) \end{array} \right\}$$

$H_2^R(X)$ is isomorphic to $\mathbb{Z}^{k^2}$

$(1,1)$	$\cdots$	$(1,i_1-1)$	$\cdots$	$(1,i_{k-3})$	$\cdots$	$(1,n-2)$		
$\vdots$	$\ddots$	$\vdots$	$\cdots$	$\vdots$	$\ddots$	$\vdots$	$\sum_{x=1}^{i_1-1}(x,n-1)$	$-(1,n-1)-(2,n-1)-\cdots$
$(i_1-1,1)$	$\cdots$	(i_1-1,i_1-1)	$\cdots$	(i_1-1,i_{k-3})	$\cdots$	$(i_1-1,n-2)$		$-(i,n-1)+(1,n)$
$(i_1,1)$	$\cdots$	(i_1,i_1-1)	$\cdots$	(i_1,i_{k-3})	$\cdots$	$(i_1,n-2)$		
$\vdots$	$\ddots$	$\vdots$	$\cdots$	$\vdots$	$\ddots$	$\vdots$	$\sum_{x=i_1}^{i_2-1}(x,n-1)$	$-(i_1,n-1)-(i_1+1,n-1)-\cdots$
$(i_2-1,1)$	$\cdots$	(i_2-1,i_1-1)	$\cdots$	(i_2-1,i_{k-3})	$\cdots$	$(i_2-1,n-2)$		$-(i_1+i-1,n-1)+(1,n)$
$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$	$\vdots$	$\vdots$		$\vdots$
$(i_{k-3},1)$	$\cdots$	(i_{k-3},i_1-1)	$\cdots$	(i_{k-3},i_{k-3})	$\cdots$	$(i_{k-3},n-2)$		
$\vdots$	$\ddots$	$\vdots$	$\cdots$	$\vdots$	$\ddots$	$\vdots$	$\sum_{x=i_{k-3}}^{n-2}(x,n-1)$	$-(i_{k-3},n-1)-(i_{k-3}+1,n-1)-\cdots$
$(n-2,1)$	$\cdots$	$(n-2,i_1-1)$	$\cdots$	$(n-2,i_{k-3})$	$\cdots$	$(n-2,n-2)$		$-(i_{k-3}+i-1,n-1)+(1,n)$
$(n-1,1)$	$\cdots$	$(n-1,i_1-1)$	$\cdots$	$(n-1,i_{k-3})$	$\cdots$	$(n-1,n-2)$	$(n-1,n-1)$	$(n-1,n)$
$(n,1)$	$\cdots$	(n,i_1-1)	$\cdots$	(n,i_{k-3})	$\cdots$	$(n,n-2)$	$(n,n-1)$	(n,n)

.

$\square$

Example

In the following quandle X, the permutation in the last column is the square of the permutation in 6−th column, and hence X is a quandle.

$*$	1	2	3	4	5	6	7
1	1	1	1	1	1	2	3
2	2	2	2	2	2	3	1
3	3	3	3	3	3	1	2
4	4	4	4	4	4	5	4
5	5	5	5	5	5	4	5
6	6	6	6	6	6	6	6
7	7	7	7	7	7	7	7

[1,1,1,1,1,1,(123)(45)(6)(7),(132)(4)(5)(6)(7)]

By the previous theorem, one can see that $H_2^R(X)$ is torsion-free, and hence $H_2^R(X) = \mathbb{Z}^{4^2} = \mathbb{Z}^{16}$, because $\sigma = (123)(45)(6)(7)$ consists of 4 disjoint cycles. Indeed, $H_2^D(X) = \mathbb{Z}^4$, $H_2^R(X) = \mathbb{Z}^{4^2} = \mathbb{Z}^{16}$ and $H_2^Q(X) = \mathbb{Z}^{12}$.

Questions

1. For two finite quandles $(Q_1, *_1)$ and $(Q_2, *_2)$, let $(X, *)$ be the quandle whose operation table is given as follows.

Q_1	id
id	Q_2

   Does the equality $torH_n^R(X) = torH_n^R(Q_1) \oplus torH_n^R(Q_2)$ hold for $n \geqslant 4$?

2. Let X be the quandle whose operation table is one of the followings.

$1 \cdots 1$	τ
$1 \cdots 1$	1

or

$1 \cdots 1$	$\tau \ \tau^i$
$1 \cdots 1$	$1 \ \ 1$

   Is $H_n^R(X)$ torsion-free for $n \geqslant 3$?

Thank you for your attention.